Needham, Chapter 5, Exercise 27, p. 264.
G. Palmer, May 11, 2016

Out[2939]=

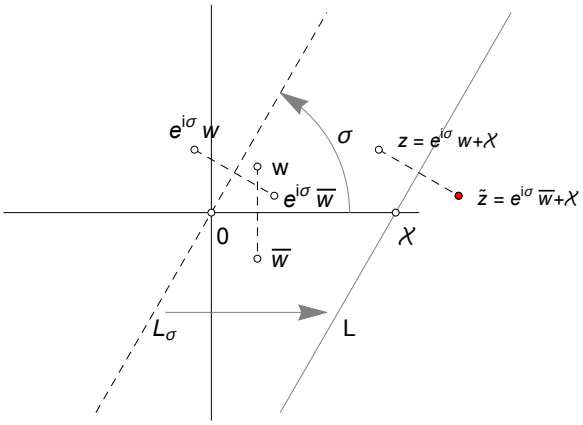



Figure 1. Schwarzian reflection on L

*(i) If L is a line passing through the real point $\chi$, and making an angle $\sigma$ with the horizontal, then show that its Schwartz function is:*

$$\mathcal{S}_L = z\, e^{-i2\sigma} + \chi\left(1 - e^{-i2\sigma}\right).$$

From p. 255, the Schwartz function is

$$z \in K \Longrightarrow \mathcal{S}_K(z) = \overline{z}.$$

This is a rotation of z by $-2\sigma$ around $\chi$. We rewrite this as a rotation centered at 0 and then translated to $\chi$ as explained in Chapter 1, Section 3, p. 18. Since z lies on L, the number (vector) to be rotated is $(z-\chi)$. We translate the center of rotation from $\chi$ to the origin, rotate $(z-\chi)$ by $-2\sigma$, and translate back to $\chi$. Note! We use the script $\mathcal{R}$ for rotation and the gothic $\mathfrak{R}$ for reflection.

$$\mathcal{R}_\chi^{-2\sigma}(z) = T_\chi \circ \mathcal{R}_0^{-2\sigma} \circ T_{-\chi}(z)$$

$$= e^{-i2\sigma}(z-\chi) + \chi$$

$$= e^{-i2\sigma} z - e^{-i2\sigma}\chi + \chi$$

$$= z\, e^{-i2\sigma} + \chi\left(1 - e^{-i2\sigma}\right)$$

$$= \mathcal{S}_L(z)$$

QED

Vasco takes a different and probably more appropriate approach, given the information presented in

this chapter. In Ch. 5, Sect. 5, p. 255, we find $\mathcal{S}'_K = e^{-2\theta}$. Applied to this problem, we write

$$\mathcal{S}'_L(z) = e^{-i2\sigma}$$

Then, integrating in terms of z,

$$\mathcal{S}_L(z) = \int \mathcal{S}'_L(z)\, dz = \int e^{-i2\sigma}\, dz = z e^{-i2\sigma} + c$$

At $\mathcal{X}$,

$$\mathcal{S}_L(\mathcal{X}) = \mathcal{X} e^{-i2\sigma} + c = \mathcal{X}$$

Then

$$c = \mathcal{X} - \mathcal{X} e^{-i2\sigma} = \mathcal{X}\left(1 - e^{-i2\sigma}\right)$$

$$\mathcal{S}_L(z) = z e^{-i2\sigma} + \mathcal{X}\left(1 - e^{-i2\sigma}\right)$$

QED

*(ii) If C is a circle with centre p and radius r, show that its Schwarz function is*

$$S_C(z) = \overline{p} + \frac{r^2}{z - p}$$

The Schwartz function is

$z \in K \Longrightarrow \mathcal{S}_K(z) = \overline{z}$.

$r = |z-p|$

$(z-p)\,(\overline{z - p}) = |z - p|^2 = r^2$

$$\overline{z - p} = \frac{r^2}{z - p}$$

$$\overline{z} = \overline{p} + \frac{r^2}{z - p} \quad \text{(because } \overline{z - p} = \overline{z} - \overline{p}\text{)}$$

QED

*(iii) Verify the claim that in both these cases $z \mapsto \mathcal{R}(z)$ is the ordinary reflection, even if z is far from the curve.*

Here I follow Vasco's solution:

Select plot a point w at any distance from the real line. Then $\overline{w}$ represents an ordinary reflection of w about the real line. Rotate the ℂ plane with a copy of the real line by $\sigma$ to create the dotted line $L_\sigma$, which passes through the origin with angle $\sigma$. The two points have been rotated and they are now are symmetrical on $L_\sigma$. They are described by $e^{i\sigma}\, w$ and $e^{i\sigma}\, \overline{w}$. Now translate L by $\chi$. The two points become

$$z = (e^{i\sigma}\, w + \chi)$$

$$\tilde{z} = (e^{i\sigma}\, \overline{w} + \chi)$$

We determined in (i) that $S_L(z) = z\, e^{-i2\sigma} + \chi\left(1 - e^{-i2\sigma}\right)$.

Then

$$\mathcal{R}_L(z) = \overline{S_L(z)} = \overline{z\, e^{-i2\sigma} + \chi\left(1 - e^{-i2\sigma}\right)}$$

$$= \overline{z}\, e^{i2\sigma} + \chi\left(1 - e^{i2\theta}\right) \qquad (1)$$

With $\mathcal{R}_L(z)$, we ought to be able to reflect $z = (e^{i\sigma}\, w + \chi)$ across L to the point $\tilde{z} = e^{i\sigma}\overline{w} + \chi$. The result of plotting using equation (1) also appears to be a reflection as shown by the red dot in Figure 1.

$$\mathcal{R}_L(e^{i\sigma}\, w + \chi) = (\overline{e^{i\sigma}\, w + \chi})\, e^{i2\sigma} + \chi\left(1 - e^{i2\sigma}\right)$$

$$= e^{i2\sigma} e^{-i\sigma}\overline{w} + e^{2\,i\sigma}\chi + \chi - e^{i2\sigma}\chi$$

$$= e^{i\sigma}\overline{w} + \chi$$

Then a reflection of the reflection should produce an identity.

$$\mathcal{R}_L \circ \mathcal{R}_L(z) = \mathcal{R}_L(e^{i\sigma}\, \bar{w} + \chi)$$

$$= (\overline{e^{i\sigma}\, \overline{w} + \chi})\, e^{i2\sigma} + \chi\left(1 - e^{i2\sigma}\right)$$

$$= e^{i2\sigma} e^{-i\sigma} w + e^{2\,i\sigma}\chi + \chi - e^{i2\sigma}\chi$$

$$= e^{i\sigma} w + \chi$$

$$= z$$

So it appears that $\mathcal{R}_L(z)$ is just the ordinary reflection in a line. I don't think we have seen an equation for the complex reflection in a line prior to this, so it should be useful.

In (ii), the reflection in a circle,

$S_C(p) = \overline{p} + \frac{r^2}{z-p}$ is given. (2)

The RHS of (2) is equivalent to $I_C(\overline{p})$.

$S_C$(p) is the conjugate of $I_C(p)$. Then $\Re_C$(p) = $\overline{S_C(p)}$ = $I_C(p)$, which is “the ordinary reflection” in a circle.