Needham, Chapter 5, Exercise 26, p. 264.
G. Palmer, May 29, 2016

26. Unlike inversion in a circle, show that Schwarzian reflection in an ellipse E (see figure [31]) does not interchange the interior and the exterior. Indeed, how does $\mathcal{R}_E(z)$ behave for large values of |z|?

We now see why on p. 256, Needham provided only the numeric form of $\mathcal{R}_E$. He was leaving the symbolic derivation for this question, or the question has a numerical solution.

If $\mathcal{R}$ exchanges the inside for the outside, then $\mathcal{R}_E \circ \mathcal{R}_E(z) = z$.

A single contradictory example is enough to show that *an ellipse E does not interchange the interior and the exterior.* Is it the case with the example on p. 256?

$$\mathcal{R}_E(z) = \frac{1}{3}(5\bar{z} - 4\sqrt{\bar{z}^2 - 3})$$

Try a point outside the ellipse, such as the point 2.5 + 1.6 i.

$\mathcal{R}_E(\mathcal{R}_E(2.5 + 1.6\,i)) = 2.11 - 1.23$ i (See the *Mathematica* cell, below.)

We see that $\mathcal{R}_E(\mathcal{R}_E(2.5 + 1.6\,i)) \neq 2.5 + 1.6\,i$. Since the test for exchange of interior and exterior points fails in this case, we conclude that Schwarzian reflection in an ellipse E does not in general interchange the interior and the exterior.

In[2674]:=
```
Clear["Global`*"];

R[z_] := (1 / 3) (5 Conjugate[z] - 4 Sqrt[Conjugate[z]^2 - 3]);

R[R[2.5 + 1.6 I]]
```

Out[2676]=
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2.10878 - 1.22527 i
```

How does $\mathcal{R}_E(z)$ behave for large values of |z|?

For large values of |z|, $\mathcal{R}_E(z)$ falls outside the ellipse, so there can be no exchange between the inside and the outside. We can be sure of this when $|\mathcal{R}_E(z)| = |\frac{1}{3}(5\bar{z} - 4\sqrt{\bar{z}^2 - 3})| > 2$, where 2 is the value of the largest semi-axis. We can see in particular that when x is very large, where z = x + iy, $\mathcal{R}_E(z)$ falls outside the ellipse when |z| > 6.