Needham, Chapter 5, Exercise 25, p. 264.
G. Palmer, April 29, 2016

Out[41]=

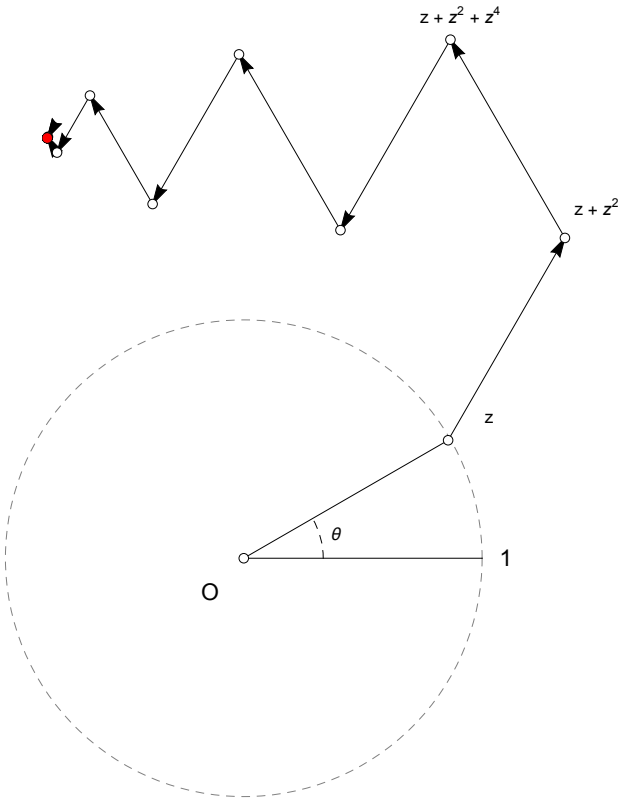


Fig. 1. f(z) = z + z^2 + z^4 + z^8 + z^{16} + ... + $z^{2\wedge n}$

θ: Pi/6, |z|: 0.99, n: 20

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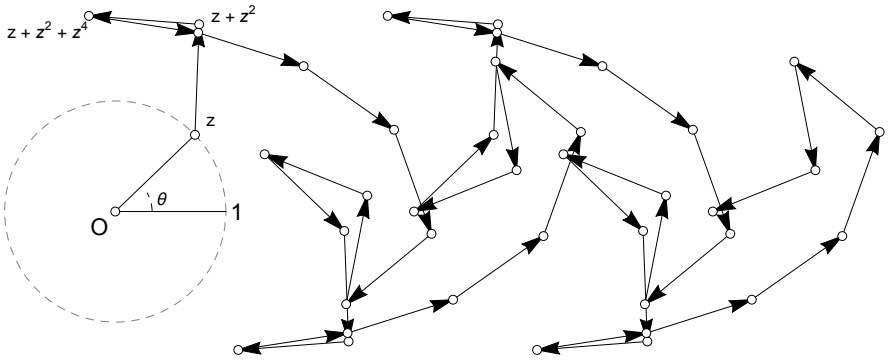


Fig. 2. f(z) = z + z^2 + z^4 + z^8 + z^{16} + ... + $z^{2\text{^}n}$

θ: 0.766242, |z|: 1., n: 40

In Figure 1, with |z| < 1, the red point is the convergence point at $z^{2\text{^}40}$. In Figure 2, with r = 1, f(z) shows no signs of converging.

25. Sometimes the circle of convergence of a power series is so densely packed with singularities that it becomes a genuine barrier for the geometric mapping beyond which it cannot be continued. This is called a natural boundary. An example of this is furnished by

$$f(z) = z + z^2 + z^4 + z^8 + z^{16} + \ldots,$$

which converges inside the unit circle. Show that every point of |z| = 1 is either a singularity itself, or else has singularities arbitrarily near to it. [Hint: What is f(1)? Now note that f(z) = z + f(z^2), and deduce that f is singular when z^2 = 1. Continuing in this manner, show that the 2^n-th roots of unity are all singular.]

A singularity of f is a point at which f is not analytic, but every neighborhood of the point contains at least one point at which f is analytic (Mathews and Howell, *Complex Analysis for Physics and Engineering*, 5th Ed.).

By the ratio test, $\lim_{n\to\infty}|\frac{c_n}{c_{n+1}}| = |\frac{1}{1}|$, confirming that f converges for z inside the unit circle.
From the Hint, f(1) = ∞. So f apparently goes to infinity when |z| ≥ 1.

Does f(z) have a derivative at z when |z| = 1?

$$f'(z) = 1 + 2z + 4z^3 + 8z^7 + 16z^6 + \ldots.$$

$$f'(1) = 1 + 2 + 4 + 8 + 16 + \ldots$$

There is a derivative, which also goes to infinity at z = 1, so this is not a point that could be part of a

smooth curve. What about points in the neighborhood of 1?

Let $z = re^{i\theta}$ and let r = 1.

$$f(z) = e^{i\theta} + e^{i2\theta} + e^{i4\theta} + e^{i2^{n\theta}} + ...$$

$$= \sum_{n=1}^{\infty} e^{i2^n \theta}$$

This way of writing f(z) makes it evident that as $n \longrightarrow \infty$, $|f(c)| \geq 1$ and f winds to an infinitely large angle.

Now note that $f(z) = z + f(z^2)$, *and deduce that f is singular when* $z^2 = 1$.

$$f(z^2) = z^2 + (z^2)^2 + (z^2)^4 + (z^2)^8 + (z^2)^{16} + ...,$$

$$= 1 + (1^2)^2 + (1^2)^4 + (1^2)^8 + (1^2)^{16} + ...$$

$$= 1 + 1 + 1 + 1 + 1 + ...$$

$f(z) \longrightarrow \infty$, so it is singular.

Continuing in this manner, show that the 2^n*-th roots of unity are all singular.*

The roots of unity are the roots of $z^n - 1$. The 2^n*-th* roots of unity are the roots of $(z^{2^n}-1)$.

$$f(z^{2^n}) = z^{2^n} + f(z^{2^n}) = 1 + \infty = \infty$$

Now return to the question: *Show that every point of* $|z| = 1$ *is either a singularity itself, or else has singularities arbitrarily near to it.*

Here we attempt to reproduce the answer provided by Vasco:

Let us say that we wish to find a singularity at a distance from p that is less than an infinitesimal distance ϵ, which can be arbitrarily small. Pick a point p anywhere on the unit circle. Then if p is a singularity, pick another point. If this point is not a singularity pick a 2-nth root of unity on either side of p. These two points are guaranteed to be singular by the previous answer. One of them, say rMax, should be chosen to be farther away from p than the other. Find a third 2-nth root of unity midway between the two previous roots. We should have no problem locating one, because there are an infinite number of such roots, spaced infinitesimally closely. Discard rMax and replace it with the third root to create an interval containing p that is smaller than the previous interval. Continue finding 2-nth roots of unity at the halfway point and discarding the most distant root until a root is found that is closer than ϵ.

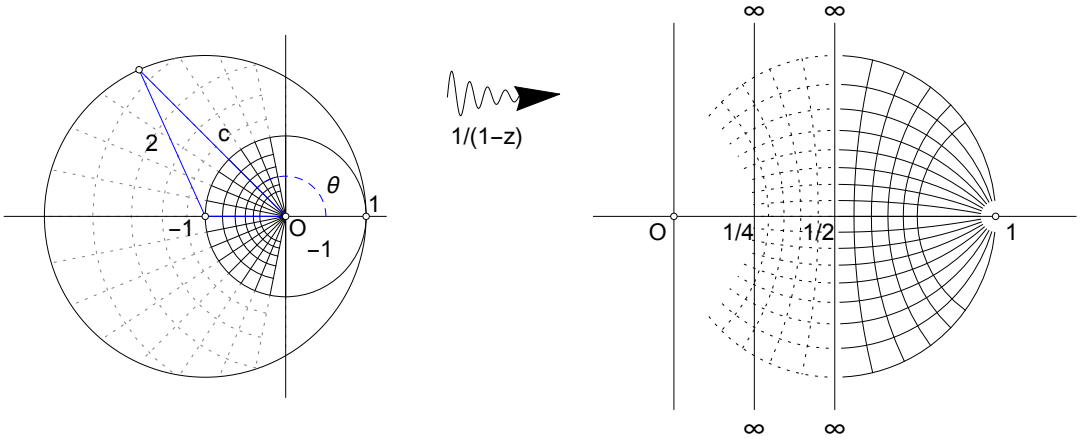


Fig. 3. Mapping with converging functions.
After Needham, Fig. [25], p. 248.

In XI 1, Introduction, on p. 248, paragraph 2, Needham wrote that

1/(1-z) agrees with both G and H within their circles of convergence; it thus constitutes the complete analytic continuation of the mapping. [We encourage you to use this fact to check the details of the figure.]

Figure 3 uses the cosine rule and the fact that the radius of the larger circle is 2 to find the length of the rays that extend from the origin to the larger circle. If c is the length of a ray and θ the angle of the ray, then by the cosine rule,

$$2^2 = 1^2 + c^2 - (2)(1)(c)\cos(\text{Pi}-\theta)$$

$$c^2 - 2\cos(\text{Pi}-\theta)\, c - 3 = 0$$

which is remarkably similar to the quadratic equation for the product of the conjugate pair of roots of unity in DeCartes' Factor Theorem! i.e. $z^2 - 2z\cos\left[\frac{2k\pi}{n}\right] + 1$.

The quadratic equation yields

$$c = \cos(\text{Pi}-\theta) \pm \left(\cos^2(\text{Pi} - \theta) + 3\right)^{1/2}$$

Then the rays can be mapped as points $re^{i\theta}$, where $r = c(\theta)$ and θ is varied by Pi/16. The arcs are mapped using $\text{Pi}/2 \le \theta \le 3\text{Pi}/2$. The ten arcs have radii described by

$r_n = .8\, r_{n-1}$, where $r_0 = 3$.

Using

$$G(z) = 1 + z + z^2 + z^3 + ...,$$

we were able to map the area inside the unit circle on the LHS to the area between 1/2 and 1 on the RHS as shown in Figure [25], p. 248. Using

$$H(z) = H(z) = \frac{1}{2}\left[1 + \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 + ...\right]$$

we mapped the areas on both the inside and the outside of the unit circle on the LHS to the RHS between 1/2 and 1 and between 1/4 and 1/2, respectively as shown in [25]. We then used 1/(1-z) to obtain the identical mapping shown in Figure 3.