Needham, Chapter 5, Exercise 24, p. 264.
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(1) Let the position at time t of a moving particle in ℂ be $z(t) = r(t)\, e^{i\theta(t)}$.

(i) Show that the acceleration of the particle is

$$\ddot{z} = [r - r\dot{\theta}^2]\, e^{i\theta} + [2\dot{r}\dot{\theta} + r\ddot{\theta}]\, ie^{i\theta}.$$

$$\dot{z} = \dot{r}\, e^{i\theta} + r\left(e^{i\theta}\right)' = \dot{r}\, e^{i\theta} + rie^{i\theta}\dot{\theta} = \dot{r}\, e^{i\theta} + i\dot{\theta}z$$

$$\ddot{z} = (\dot{r}\, e^{i\theta})' + (i\dot{\theta}z)'$$

$$= \left(\ddot{r}\, e^{i\theta} + \dot{r}\, ie^{i\theta}\dot{\theta}\right) + (i\ddot{\theta}z + i\dot{\theta}\dot{z})$$

$$= \left(\ddot{r}\, e^{i\theta} + \dot{r}\, ie^{i\theta}\dot{\theta}\right) + (i\ddot{\theta}re^{i\theta} + i\dot{\theta}(\dot{r}\, e^{i\theta} + i\dot{\theta}z))$$

$$= = \left(\ddot{r}\, e^{i\theta} + \dot{r}\, ie^{i\theta}\dot{\theta}\right) + (i\ddot{\theta}re^{i\theta} + i\dot{\theta}(\dot{r}\, e^{i\theta} + i\dot{\theta}re^{i\theta}))$$

$$= \left(\ddot{r}\, e^{i\theta} - \dot{\theta}^2 re^{i\theta}\right) + (\dot{r}\, ie^{i\theta}\dot{\theta} + i\ddot{\theta}re^{i\theta} + i\dot{\theta}\dot{r}\, e^{i\theta})$$

$$= (\ddot{r} - \dot{\theta}^2 r)e^{i\theta} + i(\dot{r}\dot{\theta} + r\ddot{\theta} + \dot{\theta}\dot{r})\, e^{i\theta}$$

$$= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\, ie^{i\theta}$$

(ii) What are the radial and transverse components of the acceleration?

The radial component is a vector in the same direction as the particle, so it is $(\ddot{r} - r\dot{\theta}^2)e^{i\theta}$. The transverse component is a vector orthogonal to the direction of the particle, so it is $(2\dot{r}\dot{\theta} + r\ddot{\theta})\, ie^{i\theta}$.

(iii) If the particle is moving in a central force field, with the centre of force at the origin, deduce that the areal speed $\mathcal{A} = \left(r^2\dot{\theta}/2\right)$ *is constant....*

We can show that the areal speed is constant by showing that the areal acceleration equals zero. The areal acceleration is equal to the magnitude of the transverse component.

Since the center of force is at the origin, all the acceleration of the particle points at the origin. It is all in the radial component. That means that the transverse component is zero. *Ipso facto*, $\mathcal{A} = \left(r^2\dot{\theta}/2\right)$ *is constant*. But we can demonstrate this algebraically.

$$(2\dot{r}\dot{\theta} + r\ddot{\theta})\, ie^{i\theta} = 0$$

Since $ie^{i\theta}$ is never zero, we can write

$$(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \qquad (1)$$

The areal acceleration is

$$\dot{\mathcal{A}} = \left(r^2\dot{\theta}/2\right)'$$

$$= (1/2)\,(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

$$= (1/2)r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0, \text{ by substitution from (1)}$$

Since the areal acceleration is 0, the areal speed must be constant.