Neeham, Chapter 5, Exercise 23, pp. 263-4.
G. Palmer, April 24, 2:45 pm

23. As another continuation of Ex. 20, let us see how the Schwarzian derivative {f(z), z} of Ex. 19 arises rather naturally in the context of curvature.

(1) Show that

$$\frac{d\theta}{dt} = \frac{1}{|f'|} Im\{f(z), z\}$$

Useful preliminaries:

$\{f(z),z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$, from Ex. 19, p. 262

We are trying to derive

$$\frac{1}{|f'|}\mathrm{Im}\left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right) \qquad (1)$$

We will need

$$\frac{d|v|}{dt} = \frac{d(v\bar{v})^{\frac{1}{2}}}{dt}$$

$$= \frac{1}{2|v|}(v\dot{\bar{v}} + \dot{v}\bar{v}) = \frac{1}{2}\left(\frac{v\dot{\bar{v}}}{|v|} + \frac{\dot{v}\bar{v}}{|v|}\right) \qquad (2)$$

$\tilde{\kappa} = \frac{\mathrm{Im}(\dot{v}/v)}{|v|}$, from Ex. 20, (i)

We should recognize that $(\mathrm{Im}(\dot{v}/v))' \equiv \mathrm{Im}\left(\left(\frac{f''}{f'}\right)'\right)$, which is the first imaginary term of (1). Once we obtain $\mathrm{Im}(\dot{v}/v)$, it seems good practice to try to preserve it. We have it already in Ex. 20(i).

$|v|\tilde{\kappa} = \mathrm{Im}(\dot{v}/v)$, multiplying both sides by $|v|$

Differentiate both sides.

$$\frac{d|v|}{dt}\tilde{\kappa} + |v|\frac{d\tilde{\kappa}}{dt} = \frac{d\,\mathrm{Im}(\dot{v}/v)}{dt}$$

Rearrange.

$$|v|\frac{d\tilde{\kappa}}{dt} = \frac{d\,\mathrm{Im}(\dot{v}/v)}{dt} - \frac{d|v|}{dt}\tilde{\kappa}$$

$$= \mathrm{Im}\frac{d(\dot{v}/v)}{dt} - \frac{d|v|}{dt}\frac{\mathrm{Im}(\dot{v}/v)}{|v|}$$

We now have the first imaginary term of (1), so we have only to work out the second imaginary term of (1) and multiply through by $\frac{1}{|v|}$.

Since $\frac{d|v|}{dt}$ is real and Im(z + w) = Im(z) + Im(w), we can write

$$\frac{d\tilde{\kappa}}{dt} = \frac{1}{|v|}\text{Im}\left(\frac{d(\dot{v}/v)}{dt} - \frac{d|v|}{dt}\frac{(\dot{v}/v)}{|v|}\right), \text{ also multiplying both sides by } \frac{1}{|v|} \qquad (3)$$

$$= \frac{1}{|v|}\text{Im}\left(\frac{d(\dot{v}/v)}{dt} - \frac{1}{2}\left(\frac{v\dot{\bar{v}}}{|v|} + \frac{\dot{v}\bar{v}}{|v|}\right)\frac{\dot{v}}{v|v|}\right), \text{ by substitution from (2)} \qquad (4)$$

$$= \frac{1}{|v|}\text{Im}\left(\frac{d(\dot{v}/v)}{dt} - \frac{1}{2}\left(\frac{v\dot{\bar{v}}\dot{v}}{|v|^2 v} + \frac{\dot{v}^2\bar{v}}{v(v\bar{v})}\right)\right), \text{ by expansion and rewriting } |v|^2$$

$$= \frac{1}{|v|}\text{Im}\left(\frac{d(\dot{v}/v)}{dt} - \frac{1}{2}\left(\frac{|\dot{v}|^2}{|v|^2} + \frac{\dot{v}^2}{v^2}\right)\right), \text{ by cancelation of } \frac{z}{z} \text{ and rewriting } (\dot{\bar{v}}\,\dot{v})$$

$$= \frac{1}{|v|}\text{Im}\left(\frac{d(\dot{v}/v)}{dt} - \frac{1}{2}\left(\frac{\dot{v}}{v}\right)^2\right), \text{ because } \frac{|\dot{v}|^2}{|v|^2} \text{ is real, it can be deleted}$$

$$\equiv \frac{1}{|f'|}\text{Im}\left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right)$$

$$= \frac{1}{|f'|}\,Im\{f(z), z\}$$

(ii) Use Ex. 19, part (iv) to deduce that if f(z) is a Möbius transformation then $\frac{d\tilde{\kappa}}{dt} = 0$. Why is this result geometrically obvious.

From 19(iv) all Möbius transformations have vanishing Schwartzian derivatives, so

$$\frac{d\tilde{\kappa}}{dt} = \frac{1}{|f'|}\text{Im}\left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right) = \frac{1}{|f'|}\text{Im}(0) = 0$$

The result is geometrically obvious, because Möbius transformations map lines to lines or circles and circles to circles or lines. In solving Ex. 23 (i), we used Ex. 20 and we set v = f(t). This implies that f in the Schwartzian derivative is f(t). Since t is linear, a Möbius transformation must map it to a line or a circle. A line has $\tilde{k} = 0$, so $\frac{d\tilde{\kappa}}{dt} = 0$. A circle has $\tilde{k} = \text{const.}$, so again, $\frac{d\tilde{\kappa}}{dt} = 0$.