Needham, Chapter 5, Exercise 22, p. 263.
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22. Continuing from Ex. 20, take f(z) to be a general Möbius transformation:

$$w = f(t) = \frac{at + b}{ct + d}$$

where Δ = (ad -bc) ≠ 0. Show that the curvature of this path is

$$\tilde{\kappa} = \left| \left(\frac{2c^2}{\Delta}\right) Im\left(\frac{d}{c}\right) \right|,$$

in agreement with Ex. 18, p. 186. The fact that his is constant provides a new proof that the image is a circle, for only circles have constant curvature.

Rewrite f(t) in terms of (3), p. 124:

$$t \longrightarrow t + \frac{d}{c}$$

$$t + \frac{d}{c} \longrightarrow \frac{1}{t+\frac{d}{c}}$$

$$\frac{1}{t+\frac{d}{c}} \longrightarrow -\frac{(ad-bc)}{c^2} \frac{1}{t+\frac{d}{c}} = -\frac{\Delta}{c^2} \frac{1}{t+\frac{d}{c}}$$

$$-\frac{\Delta}{c^2} \frac{1}{t+\frac{d}{c}} \longrightarrow -\frac{\Delta}{c^2} \frac{1}{t+\frac{d}{c}} + \frac{a}{c}$$

$$f(t) = w = -\frac{\Delta}{c^2} \frac{1}{t+\frac{d}{c}} + \frac{a}{c}$$

$$v = f'(t) = \left(-\frac{\Delta}{c^2} \frac{1}{t+\frac{d}{c}} + \frac{a}{c}\right)'$$

$$= \frac{\Delta}{c^2} \frac{1}{\left(t+\frac{d}{c}\right)^2}$$

$$\dot{v} = -\frac{2\Delta}{c^2} \frac{1}{\left(t+\frac{d}{c}\right)^3}$$

From 20 (1)

$$\tilde{\kappa} = \frac{Im(\dot{v}/v)}{|v|}$$

By substitution

$$\tilde{K} = \frac{\text{Im}\left(-\frac{2\Delta}{c^2}\frac{1}{\left(t+\frac{d}{c}\right)^3}\Big/\frac{\Delta}{c^2}\frac{1}{\left(t+\frac{d}{c}\right)^2}\right)}{\left|\frac{\Delta}{c^2}\frac{1}{\left(t+\frac{d}{c}\right)^2}\right|}$$

$$= \frac{\text{Im}\left(-2\frac{1}{\left(t+\frac{d}{c}\right)}\right)}{\left|\frac{\Delta}{c^2}\right|\left|\frac{1}{\left(t+\frac{d}{c}\right)^2}\right|} = -2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\frac{\frac{1}{\left(t+\frac{d}{c}\right)}}{\left|\frac{1}{\left(t+\frac{d}{c}\right)^2}\right|^2}\right]$$

$$= -2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\frac{1}{\left(t+\frac{d}{c}\right)}\left|t+\frac{d}{c}\right|^2\right]$$

$$= -2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\frac{\left(t+\frac{d}{c}\right)\overline{\left(t+\frac{d}{c}\right)}}{\left(t+\frac{d}{c}\right)}\right]$$

$$= -2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\overline{\left(t+\frac{d}{c}\right)}\right]$$

$$= -2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\overline{\frac{d}{c}}\right] = 2\left|\frac{c^2}{\Delta}\right|\text{Im}\left[\frac{d}{c}\right]$$

To match Needham's derivation, we take only the positive value of $\tilde{\kappa}$:

$$\tilde{\kappa} = \left|2\frac{c^2}{\Delta}\text{Im}\left[\frac{d}{c}\right]\right|$$