Needham, Chapter 5, Exercise 21, p. 263.
G. Palmer, April 20, 2016

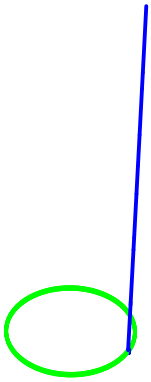


Fig. 1. a and ω fixed,
b = .05 (green), b = 1000 (blue)

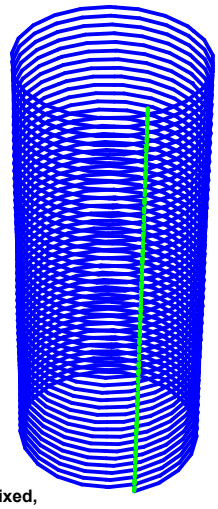


Fig. 2. a and b fixed,
ω = 0.05 (green), ω = 300 (blue)

21. In 3-dimensional space, let (X, Y, Z) be the coordinates of a moving particle. If X = a cos ωt, Y = a sin ωt, and Z = bt, then the path traced by the particle is a helix.

(i) Give interpretations for the numbers a, ω, and b.

a is the radius
ω is the speed of winding
b is the length of the central axis of the helix

(ii) If a and ω remain fixed, what does the helix look like in the two limiting cases of b becoming very small or very large?

b very small: circle or arc; $\tilde{\kappa}$ = 1/r = 1/a.
b very large: straight line orthogonal to radius; $\tilde{\kappa}$ = 0

What if a and b remain fixed while ω becomes very small or very large?

ω very small: straight line orthogonal to radius; $\tilde{\kappa}$ = 0
ω very large: cylinder; $\tilde{\kappa}$ = 1/r = 1/a

(iii) What limiting values would you anticipate for the curvature of the helix for each of the limiting cases considered in (ii).

See (ii).

(iv) Use Ex. 20(iv) to show that the curvature of the helix is

$$\tilde{\kappa} = \frac{a\omega^2}{b^2 + a^2\,\omega^2}$$

I did this one with brute force. Vasco has a more clever derivation.

$$\mathrm{h} = \begin{pmatrix} a\cos\omega t \\ a\sin\omega t \\ bt \end{pmatrix}$$

$$\mathrm{v} = \begin{pmatrix} -a\omega\sin\omega t \\ a\omega\cos\omega t \\ b \end{pmatrix}, \quad \dot{v} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -a\omega^2\sin\omega t \\ 0 \end{pmatrix}$$

$$|\mathbf{v}\times\dot{\mathbf{v}}| = \begin{pmatrix} v_2\,\dot{v}_3 - v_3\,\dot{v}_2 \\ v_3\,\dot{v}_1 - v_1\,\dot{v}_3 \\ v_1\,\dot{v}_2 - v_2\,\dot{v}_1 \end{pmatrix} = \begin{pmatrix} -ab\omega^2\sin\omega t \\ -ab\omega^2\cos\omega t \\ (a\omega\sin\omega t)\,a\omega^2\sin\omega t \;-\; (a\omega\cos\omega t)\left(-a\omega^2\cos\omega t\right) \end{pmatrix}$$

$$= \left|\begin{pmatrix} -ab\omega^2\sin\omega t \\ -ab\omega^2\cos\omega t \\ a^2\,\omega^3\sin^2\omega t + a^2\,\omega^3\cos^2\omega t \end{pmatrix}\right|$$

$$= \left| \begin{pmatrix} -ab\omega^2 \sin \omega t \\ -ab\omega^2 \cos \omega t \\ a^2 \omega^3 \end{pmatrix} \right|$$

$$= \left| \left((-ab\omega^2 \sin \omega t)^2 + (-ab\omega^2 \cos \omega t)^2 + (a^2 \omega^3)^2 \right)^{1/2} \right|$$

$$= \left| (a^2 b^2 \omega^4 + a^4 \omega^6)^{1/2} \right| = a\omega^2 \left| (b^2 + a^2 \omega^2)^{1/2} \right|$$

$$|\boldsymbol{v}|^3 = ((a\omega)^2 + b^2)^{3/2}$$

Then, using only positive square roots, because a must be positive anyway,

$$\frac{|\boldsymbol{v} \times \dot{\boldsymbol{v}}|}{|\boldsymbol{v}|^3} = \frac{a\omega^2 (b^2 + a^2 \omega^2)^{1/2}}{(b^2 + a^2 \omega^2)^{3/2}} = \frac{a\omega^2}{b^2 + a^2 \omega^2}$$

and use this to confirm your hunches in (iii).

a and ω fixed, b small: as $b \longrightarrow 0$, $\tilde{\kappa} = \frac{a\omega^2}{b^2 + a^2 \omega^2} \longrightarrow \frac{a\omega^2}{a^2 \omega^2} \longrightarrow \frac{1}{a}$

a and ω fixed, b large: as $b \longrightarrow \infty$, $\tilde{\kappa} = \frac{a\omega^2}{b^2 + a^2 \omega^2} \longrightarrow 0$

a and b fixed, ω small: as $\omega \longrightarrow 0$, $\tilde{\kappa} = \frac{a\omega^2}{b^2 + a^2 \omega^2} \longrightarrow 0$

a and b fixed, ω large: as $\omega \longrightarrow \infty$, $\tilde{\kappa} = \frac{a\omega^2}{b^2 + a^2 \omega^2} \longrightarrow \frac{1}{a}$

These results confirm the hunches.