Needham, Chapter 5, Exercise 20, p. 262.
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Think of the real axis as representing time t, and let a moving particle w = f(t) (where f(t) is analytic) trace an orbit curve C. The velocity is then v = $\dot{w}$ = f'(t).

(i) Use (23) to show that the curvature of C is

$$\tilde{\kappa} = \frac{Im(\dot{v}/v)}{|v|}$$

$$\tilde{\kappa} = \frac{1}{|f'(p)|}(\mathrm{Im}[\frac{f''(p)\,\hat{\xi}}{f'(p)}] + \kappa) \qquad (23), \text{ p. 234}$$

We can think of t as the real line with curvature $\kappa = 0$. This means that $\hat{\xi} = 1$, so f''(p) $\hat{\xi}$ = f''(p).

Substitute

f''(p) for f''(p) $\hat{\xi}$
v for f'(t)
$\dot{v}$ for f''(t)
0 for κ

$$\tilde{\kappa} = \frac{1}{|v|}(\mathrm{Im}[\frac{\dot{v}}{v}] + 0 = \frac{\mathrm{Im}(\dot{v}/v)}{|v|}$$

QED

(ii) Argue that this result does not in fact depend on C being produced by an analytic mapping, but is instead true of any motion for which the velocity v and acceleration $\dot{v}$ are well-defined.

Under the given situation, $\kappa = 0$ and $\hat{\xi} = 1$ are always true. Hence, if v and $\dot{v}$ are well-defined complex numbers, one can calculate $\tilde{\kappa}$. An analytic function depends on f being infinitely differentiable, but derivatives of higher order than f'' are not required for this calculation. then this result is valid for any movements for which v and $\dot{v}$ are well-defined.

(iii) Show that the formula may be rewritten as

$$\tilde{\kappa} = \frac{Im(\bar{v}\,\dot{v})}{|v|^3}$$

Answer. Multiply top and bottom by $|v|^2$.

$$\tilde{\kappa} = \frac{\mathrm{Im}(\dot{v}/v)}{|v|} = \frac{|v|^2\mathrm{Im}(\dot{v}/v)}{|v|^2|v|} = \frac{(\bar{v}\,v)\,\mathrm{Im}(\dot{v}/v)}{|v|^3}$$

$$= \frac{\text{Im}\left(\left(\bar{v}\, v\right)\dot{v}\, /v\right)}{|v|^3} = \frac{Im\left(\bar{v}\,\dot{v}\right)}{|v|^3}$$

QED

(iv) Deduce that it may also be written vectorially as

$$\tilde{\kappa} = \frac{|\boldsymbol{v} \times \dot{\boldsymbol{v}}|}{|\boldsymbol{v}|^3}$$

From (20), p. 28,

$$\bar{v}\,\dot{v} = \boldsymbol{v} \bullet \dot{\boldsymbol{v}} + \text{i}\,(\boldsymbol{v} \times \dot{\boldsymbol{v}})$$

$$\text{Im}[\bar{v}\,\dot{v}] = \boldsymbol{v} \times \dot{\boldsymbol{v}}$$

$$|v| = |\mathbf{v}|$$

$$\tilde{\kappa} = \frac{\text{Im}\left(\bar{v}\,\dot{v}\right)}{|v|^3} = \frac{\boldsymbol{v} \times \dot{\boldsymbol{v}}}{|\boldsymbol{v}|^3}$$

From p. 28, we infer that $\boldsymbol{v} \times \dot{\boldsymbol{v}}$ is a real number, as

$$\boldsymbol{v} \times \dot{\boldsymbol{v}} = \left|\, \boldsymbol{v} \,\right|\left|\, \dot{\boldsymbol{v}} \,\right| \sin\left[\arg(v) - \arg(\dot{v})\right]$$

Since $\tilde{\kappa}$ is $\frac{1}{r}$, we want $\tilde{\kappa} > 0$, so

$$\tilde{\kappa} = \frac{|\boldsymbol{v} \times \dot{\boldsymbol{v}}|}{|\boldsymbol{v}|^3}$$

QED