Needham, Chapter 5, Exercise 2, p. 235.
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2. Show that the real and imaginary parts of an analytic function are harmonic, i.e. they both satisfy Laplace's equation

$$\Delta\Phi = 0$$

where Δ*, which is often instead written* ∇^2 *is defined by* $\Delta \equiv \partial_x^2 + \partial_y^2$ *and is called the* ***Laplacian****....*

C-R equations : $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$

If f = u + iv, we are to show that

$$\Delta(u) = \Delta(v) = 0$$

On a first attempt, we misread $\partial_x^2 + \partial_y^2$ to be the squares of the first partials. We believe it is also true when read that way (see below), but it is not the Laplacian. Consulting Vasco's archives, we realized that the Laplacian involves second derivatives, so we make another attempt without copying his solution directly.

$$\Delta(u) = \partial_x^2 u + \partial_y^2 u = \partial_x(\partial_x u) + \partial_y(\partial_y u)$$

$$= \partial_x(\partial_y v) + \partial_y(-\partial_x v) \quad \text{by C-R substitution}$$

$$= \partial_x(\partial_y v) - \partial_x(\partial_y v) \quad \text{by rearranging signs and reordering partials}$$

$$= 0$$

$$\Delta(v) = \partial_x^2 v + \partial_y^2 v = \partial_x(\partial_x v) + \partial_y(\partial_y v)$$

$$= \partial_x(-\partial_y u) + \partial_y(\partial_x u) \quad \text{by C-R substitution}$$

$$= -\partial_x(\partial_y u) + \partial_x(\partial_y u) \quad \text{by rearranging signs and reordering partials}$$

$$= 0$$

First attempt on a misreading of the problem:

If $\partial_x^2 + \partial_y^2 = 0$, then $\partial_x = \pm i\,\partial_y$. Since f is analytic, u and v must also be analytic, because f' is linear, i.e. f' = u' + iv'. Hence, u' and v' also have amplitwist. By the twist principle, since we know that x and y are orthogonal, ∂_x must be orthogonal to ∂_y, so $\partial_x \propto \pm i\,\partial_y$. At infinitesimal scale, $|\partial_x|$ approaches $|\partial_y|$, so $\partial_x \simeq \pm i\,\partial_y$. This applies to both u and v. It follows that $\Delta(u) = \Delta(v) = 0$.