Needham, Chapter 5, Exercise 19, p. 262.
G. Palmer, March 6, 2016

19. In more advanced work on Möbius transformations (e.g., Nehari [1952] and Beardon [1984]), an important role is played by the so-called Schwarzian derivative {f(z), z} of an analytic function f(z) with respect to z:

$$\{f(z), z\} \equiv \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

(i) Show that the Schwarzian derivative may also be written as

$$\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Answer:

Use Leibnitz' quotient rule to find $\left(\frac{f''}{f'}\right)'$.

$$\left(\frac{f''}{f'}\right)' = \frac{f'''f' - f''f''}{(f')^2}$$

$$= \frac{f'''f' - (f'')^2}{(f')^2} \qquad (1)$$

Substitute (1) into {f(z),z}.

$$\{f(z), z\} \equiv \frac{f'''f' - (f'')^2}{(f')^2} - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

$$= \frac{f'''f' - (f'')^2}{(f')^2} - \frac{(1/2)}{1}\left(\frac{f''}{f'}\right)^2$$

$$= \frac{f'''f' - (3/2)(f'')^2}{(f')^2}$$

$$= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

QED

(ii) Show that {az + b, z} = 0 = {(1/z), z}.

$$\{f(z), z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \qquad \text{given}$$

$$\{az + b, z\} = \frac{(az+b)''}{(az+b)'} - \frac{3}{2}\left(\frac{(az+b)''}{(az+b)'}\right)^2 \quad \text{by substitution into } \{f(z),z\}$$

$$\frac{(az+b)''}{(az+b)'} = \frac{0}{z} = 0$$

Hence,

$$\{az + b, z\} = 0$$

Show that $\{(1/z), z\} = 0$.

$$f' = (z^{-1})' = -z^{-2}$$

$$f'' = \left(-z^{-2}\right)' = 2z^{-3}$$

$$f''' = \left(2z^{-3}\right)' = -6z^{-4}$$

$$\{(1/z), z\} = \frac{-6z^{-4}}{-z^{-2}} - \frac{3}{2}\left(\frac{2z^{-3}}{-z^{-2}}\right)^2$$

$$= 6z^{-2} - \frac{3}{2}\left(-2z^{-1}\right)^2$$

$$= 6z^{-2} - \frac{3}{2}\left(4z^{-2}\right)$$

$$= 6z^{-2} - 6z^{-2} = 0$$

QED

(III) Let f and g be analytic functions, and write w = f(z). Show that the Schwarzian derivative of the composite function g[f(z)] = g[w] is given by the following "chain rule":

$$\{g(w), z\} = [f'(z)]^2\{g(w), w\} + \{f(z), z\}.$$

Answer:

It will be convenient to let g = g(w) and w = f and assume f = f(z).

$$g'(w) = g'f'$$

$$g''(w) = g''f'f' + g'f'' = g''(f')^2 + g'f''$$

$$g'''(w) = (g'''(f')^2 f' + g''2f'f'') + (g''f'f' + g'f''')$$

$$= g'''(f')^3 + 3g''f'f'' + g'f'''$$

$$\frac{g'''}{g'} = \frac{g''' (f')^3 + 3 g'' f' f'' + g' f'''}{g' f'}$$

$$= \frac{(f')^2 g'''}{g'} + \frac{3 g'' f''}{g'} + \frac{f'''}{f'}$$

$$\left(\frac{g''}{g'}\right)^2 = \frac{\left(g'' (f')^2 + g' f''\right)^2}{(g' f')^2}$$

$$= \frac{(g'')^2 (f')^4}{(g')^2 (f')^2} + \frac{2 g'' (f')^2 g' f''}{(g')^2 (f')^2} + \frac{(g')^2 (f'')^2}{(g')^2 (f')^2}$$

$$= (f')^2 \left(\frac{g''}{g'}\right)^2 + \frac{2 g'' f''}{g'} + \left(\frac{f''}{f'}\right)^2$$

$$\{g(w),z\} = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = \left[\frac{(f')^2 g'''}{g'} + \frac{3 g'' f''}{g'} + \frac{f'''}{f'}\right]$$

$$- \frac{3}{2}\left[(f')^2 \left(\frac{g''}{g'}\right)^2 + \frac{2 g'' f''}{g'} + \left(\frac{f''}{f'}\right)^2\right]$$

$$= \frac{(f')^2 g'''}{g'} - \frac{3}{2}(f')^2 \left(\frac{g''}{g'}\right)^2 + \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

$$= (f')^2 \left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right) + \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)$$

$$= (f')^2 \{g(w), w\} + \{f(z), z\}$$

QED

(iv) Use the previous two parts to show that all Möbius transformations have vanishing Schwarzian derivative. [Hint: Recall that the mappings in part (ii) generate (via composition) the set of all Möbius transformations.]

$\{az + b, z\} = 0 = \{(1/z), z\}$ {from (ii)}

$\{g(w),z\} = (f')^2 \{g(w), w\} + \{f(z), z\}$ {from (iii)}

Not having seen Vasco's systematic method of decomposing the transformation using the steps from (3), p. 124, I decided to include step 1 with step 2 and step 4 with step 3 because both 1 and 4 involve adding constants, which gives the format (az + b), for which we know that the Schwarzian is 0.

Let f(z) = z + d/c

$\{f(z), z\} = 0$ (from (ii), $\{az + b, z\} = 0$)

Let g(z) = 1/z. The first step of our revised composition is g(f).

g(f) = 1/f = 1/(z+d/c)

{g(f), f} = 0 (from (ii), {(1/z), z} = 0)

Let h(z) = $-\frac{(ad-bc)}{c^2} z + a/c$. The second step of the composition is h(g).

h(g(f)) = $-(ad - bc)\, g/c^2 + a/c$

{h(g), g} = 0 (from (ii), {az + c}, z} = 0)

We want the Schwarzian of h(g(f(z))).

{h(g),z} = $(g')^2$ {h(g), g} + {g(f), z} (from (iii)) [valid step?]

{g(f), z} = $(f')^2$ {g(f), f} + {f(z), z}

{h(g),z} = $(g')^2$ {h(g), g} + [$(f')^2$ {g(f), f} + {f(z), z}] (by substitution)

= $(g')^2$ *0 + $(f')^2$* 0 + 0

= 0

(v) Use the previous two parts to show that the Schwarzian derivative is "invariant under Möbius transformations", in the following sense: if M is a Möbius transformation, and f is analytic, then

{M[f(z)], z} = {f(z), z}

{M[f(z)], z} = $(f')^2${M, f} + {f, z}

= $(f')^2$ * 0 + {f, z}

= {f(z), z}