Needham, Chapter 5, Exercise 18, p. 261.
G. Palmer, March 2, 2016

Let two curves C_1 and C_2 emerge from a point p in the same direction. Two examples are illustrated below. Although in both cases the angle at p is zero, there is a great temptation to say that the curves on the right meet at a smaller "angle" than those on the left. Any putative definition of such an "angle" Θ should (presumably) be conformally invariant: if the curves are mapped to $\tilde{C}_1$ and $\tilde{C}_2$ by a mapping f that preserves ordinary angles (i.e., an analytic mapping), then the new "angle" $\tilde{\Theta}$ should equal the old "angle" Θ.

(i) Newton [1670] attempted to define such a Θ as the difference of the curvatures of C_1 and C_2 at p: $\Theta \equiv \kappa_1 - \kappa_2$. Use (30) to show that this definition is not quite conformally invariant: $\tilde{\Theta} = \Theta/|f'(p)|$.

$$\tilde{\kappa} = \mathcal{K} \bullet \hat{\xi} + \frac{\kappa}{|f'|} \qquad (30)$$

All the the following are assumed to be at p. $\hat{\xi}$ is a complex unit vector at p pointing in the direction of both curves. I think we have to assume that the curvature of C_2 differs from that of C_1 even though their angles differ.

$$\tilde{\kappa_1} = \mathcal{K}_1 \bullet \hat{\xi} + \frac{\kappa_1}{|f'|} \qquad (1)$$

$$\tilde{\kappa_2} = \mathcal{K}_2 \bullet \hat{\xi} + \frac{\kappa_2}{|f'|} \qquad (2)$$

Note that

$$\mathcal{K}_1 = \mathcal{K}_2 = i \frac{\bar{f}''}{\bar{f}' |f'|}$$

Subtract (2) from (1)

$$\tilde{\kappa_1} - \tilde{\kappa_2} = \frac{\kappa_1 - \kappa_2}{|f'|}$$

$$\tilde{\Theta} = \Theta/|f'|$$

This is what we wanted.

Out[659]=

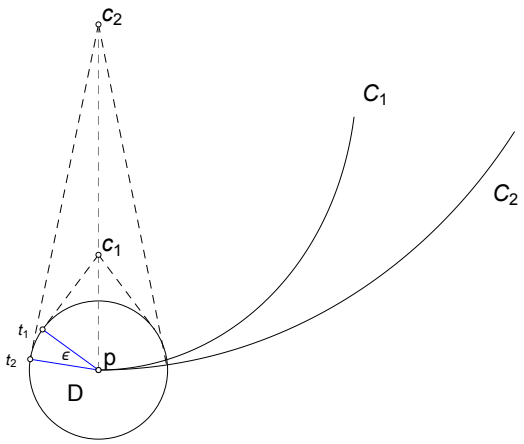


Figure 1. Angular sizes
from centers of curvature

(ii) Consider an infinitesimal disc D (radius ϵ) centred at p. Let c_1 and c_2 be centres of curvature of C_1 and C_2, and let $\mathcal{D}$ be the difference between the angular sizes of D as seen from c_1 and c_2. Show that $\mathcal{D} = \epsilon\Theta$.

Figure 1 is an attempt to represent the situation described. The centers of curvature are just guesses based on inspection of the curves. No curvature has been calculated from the curves. The tangent points are also drawn, not calculated.

$$| c_1 - p | = r_1 = 1/\kappa_1$$

$$| c_2 - p | = r_2 = 1/\kappa_2$$

$\phi_1 = \measuredangle\, p\, c_1\, t_1$, where t_1 is the contact point of a line from c_1 tangent to D.
$\phi_2 = \measuredangle\, p\, c_2\, t_2$, where t_2 is the contact point of a line from c_2 tangent to D.

An angular size of D from the viewpoint of c_1 is $(1/\kappa_1)$ $\phi_1 = \epsilon \Longrightarrow \phi_1 = \epsilon\kappa_1$
An angular size of D from the viewpoint of c_2 is $(1/\kappa_2)$ $\phi_2 = \epsilon \Longrightarrow \phi_2 = \epsilon\kappa_2$

(Compare Figure [2], p. 218. An assessment of the full size of the disc from the vantage points would require doubling of ϕ_1 and ϕ_2, but we note that on p. 240, last line, Needham wrote *Let ε denote the rate of increase of size*[5] *of $\tilde{Q}$ (again with respect to the distance it moves) as a fraction of $\tilde{Q}$'s initial size*. In the footnote, he wrote *Here we mean the linear dimensions of $\tilde{Q}$. For example, if $\tilde{Q}$ were a disc then we could take its "size" to be its radius.* It's confusing. Vasco's solution is to let ϵ be the diameter of D rather than the radius (p. c., March 2, 2016))

$$\mathcal{D} = \phi_1 - \phi_2 = \epsilon\, k_1 - \epsilon\, k_2 = \epsilon\,(\kappa_1 - \kappa_2) = \epsilon\Theta$$

(ii, cont.) If a conformal mapping f is applied to C_1, C_2, and D, deduce that $\tilde{\mathcal{D}} = \mathcal{D}$.

From (i), $\tilde{\Theta} = \Theta/|f'|$.

From the first part of (ii), $\mathcal{D} = \epsilon\Theta$. Let A be amplification.

$$\tilde{\mathcal{D}} = (A\,\epsilon)\,\tilde{\Theta} = |f'|\,\epsilon\,\Theta/|f'| = \epsilon\,\Theta = \mathcal{D}$$