Needham, Chapter 5, Exercise 16, p. 259.
G. Palmer, February 6, 2016

16. As illustrated below, a region is called convex if all of it is visible from an arbitrary vantage point inside. Let an analytic mapping f act on an origin-centred circle C to produce a simple image curve f(C), the interior of which is convex.

(i) From the formula of Ex. 15, deduce that if f maps the interior of C to the interior of f(C), then the following inequality holds at all points z of C.

$$Re\left[\frac{zf''}{f'}\right] \geq -1.$$

The formula of Ex. 15:

$$\tilde{\kappa} = \frac{1+Re\left[\frac{zf''}{f'}\right]}{|zf'|}$$

It would appear that f(C) is convex only if $\tilde{\kappa} \geq 0$ (the rate of rotation of f' wrt the distance along f(C)) at all points as z proceeds in a clockwise direction around C (which must mean that the region could be bounded by an ellipse, a circle, an oval, or any of those flattened on one side, or possibly two lines closed at the ends by segments of ellipses, circles, or ovals).

$$\frac{1+Re\left[\frac{zf''}{f'}\right]}{|zf'|} \geq 0 \Longrightarrow 1 + Re\left[\frac{zf''}{f'}\right] \geq 0 \Longrightarrow Re\left[\frac{zf''}{f'}\right] \geq -1$$

(ii) What is the analogous inequality when f maps the interior of C to the exterior of f(C). [Hint: Ex. 4, p. 211]

Ex. 4, Ch. 4, p. 211 shows that when the interior of a curve is mapped by an analytic function to the exterior of a curve, as z proceeds around the curve in a counter clockwise direction, w proceeds in a clockwise direction. Then if f(C) is convex, f' rotates in a clockwise direction and $\tilde{\kappa} \leq 0$ at all points. Then

$$\frac{1+Re\left[\frac{zf''}{f'}\right]}{|zf'|} \leq 0 \Longrightarrow 1 + Re\left[\frac{zf''}{f'}\right] \leq 0 \Longrightarrow Re\left[\frac{zf''}{f'}\right] \leq -1$$

The figures

[There is a defect in this file which corrupts any expression with typeset powers when saved to a PDF file.]

Let an analytic mapping f act on an origin-centred circle C to produce a simple image curve f(C), the interior of which is convex.

The function exp(z^2) in Figure 1 is *a* composition of two analytic functions, so it is analytic. I have renamed the unit circle and curve "z" and "w". All points on w have $\kappa > 0$, so w is in fact convex. The minimum value of κ is 0.0003. K is the circle of curvature for the point on w. Figure 2 reverses the sign of the power in f(z) = *e*^(*z*^*m*) and the sign of κ, but κ is not a vector, so I plotted it in the same direction as in Figure 2. Both figures illustrate the amplitwist of f'. The twist redirects the normal arrow to the

exterior of w.

To obtain the velocity (rate of distance on curve over θ), let z = re

w(z) = e^(z^m) = e^(re^(iθ))^m = e^{(r^m)e^(imθ)}

$\frac{dw}{d\theta}$ = e^{(r^m)e^(imθ)} (r^m) ime^(imθ)

= im(r^m)e^{(r^m)e^(imθ)}e^(imθ)

Since this is based on f(z) applied to the unit circle, r = 1 and θ = Arg(z); We let m = 2;

V = i2e^{e^(i2Arg(z))}+i2Arg(z)}

With m = 2, the velocity vector has the same angle as the blue amplitwisted tangent vector (Figure 1(b)). With m = -2, the direction of the velocity vector is reversed, as one would expect.

In[107]:=
```
(* SCRIPT FOR FIGURE 1 *)

Clear["Global`*"]

(* CONSTANTS *)

m = 2;

hues = Table[ Hue[1 / n, 1, 1, 1], {n, Range[1, 3]}];


(* FUNCTIONS *)

v[z_] := {Re @ z, Im @ z}; (* vector to complex *)


squigglyArrow := Module[{points, squigglyLn, hdPt1, head, squigArrow},

   pts = Table[{x, (.5 / (x^3)) Sin[9 Pi x]}, {x, Range[1.11, .8002 Pi , .01]}];

   squigArrow = Translate[{Arrowheads @ .06, Arrow[Line[pts]]}, {-1.1, -.01}]

  ];


point[z_, ptSz_] = {EdgeForm @ Black, FaceForm @ White, Disk[v @ z, ptSz]};

w[z_ ] := Exp[z^m];

dw[z_, m_] := m z^(m - 1) Exp[z^m];

at[z_, m_] := dw[z, m]; (* amplitwist f' *)
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ddw[z_, m_] := m (m - 1) z^(m - 2) Exp[z^m] + m^2 z^(2 (m - 1)) Exp[z^m];

k[z_, m_] := (1 + Re[z ddw[z, m] / dw[z, m]]) / Abs[z dw[z, m]] ;
(* gives curvature of w *)

wVelocity[z_, m_] := I m Exp[ Exp[I m Arg @ z] + I m Arg @ z];

SetAttributes[{v, w}, Listable];


(* GRAPHICS OBJECTS *)

(* POINTS *)

z = Exp[I 3 Pi / 4];

rϵ = .25;

ϵ = rϵ I z;

ϵi = I ϵ;


ptSz = .01;

zPoints = Table[point[pt, .02], {pt, {0, z, 1}}];

wPoints = Table[point[pt, .02], {pt, {w[z ], w[1]}}];


(* ARROWS *)

ahSz = Arrowheads @ .035; (* arrowhead size *)

ϵArrow = {Blue, ahSz, Arrow[v @ {z, z + ϵ}]};

ϵGrayArrow = {Gray, ahSz, Arrow[v @ {w @ z, w @ z + ϵ}]};

ϵiArrow = {Green, ahSz, Arrow[v @ {z, z + ϵi}]};

atϵArrow = {Blue, ahSz, Arrow[v @ {w @ z, w @ z + at[z, m] ϵ}]};

atϵiArrow = {Green, ahSz, Arrow[v @ {w @ z, w @ z + at[z, m] ϵi}]};


twistArcCmplx =
  Table[.25 Exp[I θ] + w @ z, {θ, Range[Arg @ ϵ, Arg @ (at[z, m] ϵ), .01]}];

twistArrow = {Dashed, Gray, ahSz, Arrow[v @ twistArcCmplx]};

kArrow = {Black, ahSz, Arrow[v @ {w @ z, w @ z + k[z, m] Exp[I Arg @ (at[z, m] ϵi)]}]};

velArrow = {Orange, ahSz, Arrow[v @ {w @ z, w @ z + wVelocity[z, m]}]};

arrowsA = {ϵArrow, ϵiArrow};
```

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arrowsB = {velArrow, at∈Arrow, kArrow, at∈iArrow, ∈GrayArrow, twistArrow};

sqArr = Translate[Scale[ squigglyArrow, .5], v @ 1.2];


(* CURVES *)

uCircleCmplx = Table[Exp[I θ], {θ, Range[0, 2.02 Pi, .02]}]; (* unit circle *)

(* check curvature on w curve: Table[If[k[z,m]< 0, k[z,m],""],{z,uCircleCmplx}]

   Min @ Table[ k[z,m],{z,uCircleCmplx}] *)

uCircle = {Dashed, Gray, Line[v @ uCircleCmplx]};


innerCircleCmplx[r_] := Table[ r Exp[I θ], {θ, Range[0, 2.01 Pi, .01]}];
(* 1 inner circle, complex pts *)

innerCirclesCmplx = Table[innerCircleCmplx[r / 4], {r, Range[3]}];
(* inner circles, complex pts *)

innerCircles =
 Table[{hues[[i]], Line[v @ innerCirclesCmplx[[i]]]}, {i, Range[1, 3]}];


wInnerCirclesCmplx = Table[w @ circle, {circle, innerCirclesCmplx}];
(* points, inner circles for w *)

wInnerCircles =
 Table[{hues[[i]], Line[v @ wInnerCirclesCmplx[[i]]]}, {i, Range[1, 3]}];


expCurveCmplx = Table[w @ z, {z, uCircleCmplx}];
(* complex points for w circle *)

expUnitCrv = {Gray, Dashed, Line[v @ expCurveCmplx]}; (* w circle *)



ckCntr = w @ z + Exp[I Arg[at[z, m] ∈i]];  (* circle of curvature center *)

ckCirc = {LightGray, Circle[v @ ckCntr, 1 / k[z, m]]}; (* circle of curvature *)


(* TEXT *)

default = 12;
large = 14;
small = 10;

data1 = {
   {"z", 1.1 z, default},
   {"1", 1.1, default},
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    {"0", 0 - .1 - .1 I, {default, Bold}},
    {"e^z^-2", 2 - .5 I, default},
    {"Figure 1.\nTangent vectors\nand interior mapping", -2.5 I, large},
    {"(a)", -1.5 I, default}
   };

data2 = {
    {"w", w @ z - .2, default},
    {"twist", .30 Exp[I (Arg @ (at[z, m] ϵ) - Arg @ ϵ) / 2] + w @ z - .3 - .6 I, small},
    {"κ̃ = " <> ToString[k[z, m] // N],
     w @ z + 1.1 k[z, m] Exp[I Arg @ (at[z, m] ϵi)], small},
    {"dw/dθ = " <> ToString[NumberForm[wVelocity[z, m] // N, 3]],
     w @ z + 1.2 wVelocity[z, m], small},
    {"(b)", 2.75 - 1.5 I, default},
    {"K", ckCntr + .8 Exp[-I Pi / 4], default},
    {ToString[w @ 1], w @ 1 + .2, default}
   };

textFig1 = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, data1}];

textFig2 = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, data2}];


(* COLLECTED OBJECTS *)

figObjsA = {uCircle, arrowsA, innerCircles, zPoints, textFig1, sqArr};

figObjsB = {ckCirc, expUnitCrv, arrowsB, wInnerCircles, wPoints, textFig2};

GraphicsRow[
 {
  Graphics[{figObjsA}, Background → White,
   PlotRange → {{-1.5, 2.5}, {-4, 2}}, ImageSize → Medium],
  Graphics[{figObjsB}, Background → White,
   PlotRange → {{0, 3.25}, {-4, 2}}, ImageSize → Medium]
 }
]


(* SCRIPT FOR FIGURE 2 *)

Clear["Global`*"]

(* CONSTANTS *)

m = -2;

hues = Table[ Hue[1 / n, 1, 1, 1], {n, Range[1, 3]}];
```

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(* FUNCTIONS *)

v[z_] := {Re @ z, Im @ z}; (* complex to vector *)


squigglyArrow := Module[{points, squigglyLn, hdPt1, head, squigArrow},

   pts = Table[{x, (.5 / (x^3)) Sin[9 Pi x]}, {x, Range[1, .8 Pi, .01]}];

   squigArrow = {Arrowheads @ .07, Arrow[Line[pts]]};

   Translate[squigArrow, {-1, 0}]

  ];


(* point as disk *)

point[z_, ptSz_] = {EdgeForm @ Black, FaceForm @ White, Disk[v @ z, ptSz]};

w[z_ ] := Exp[z^m];

dw[z_, m_] := m z^(m - 1) Exp[z^m];  (* w' *)

at[z_, m_] := dw[z, m]; (* amplitwist w' *)

ddw[z_, m_] := m (m - 1) z^(m - 2) Exp[z^m] + m^2 z^(2 (m - 1)) Exp[z^m]; (* w'' *)

k[z_, m_] := (1 + Re[z ddw[z, m] / dw[z, m]]) / Abs[z dw[z, m]] ; (* curvature of w *)

wVelocity[z_, m_] := I m Exp[ Exp[I m Arg @ z] + I m Arg @ z];

SetAttributes[{v, w}, Listable];


(* GRAPHICS OBJECTS *)

(* POINTS *)

z = Exp[I 3 Pi / 4];

rϵ = .25;

ϵ = rϵ I z;

ϵi = I ϵ;


ptSz = .01;

zPoints = Table[point[pt, .02], {pt, {0, 1, z}}];

wPoints = Table[point[pt, .02], {pt, {w @ z, w @ 1}}];
```

```
(* ARROWS *)

ahSz = Arrowheads @ .025; (* arrowhead size *)


ϵArrow = {Blue, ahSz, Arrow[v @ {z, z + ϵ}]};

ϵGrayArrow = {Gray, ahSz, Arrow[v @ {w @ z, w @ z + ϵ}]};

ϵiArrow = {Green, ahSz, Arrow[v @ {z, z + ϵi}]};


atϵArrow = {Blue, ahSz, Arrow[v @ {w @ z, w @ z + at[z, m] ϵ}]};

atϵiArrow = {Green, ahSz, Arrow[v @ {w @ z, w @ z + at[z, m] ϵi}]};


twistArcCmplx =
  Table[.25 Exp[I θ] + w @ z, {θ, Range[Arg @ ϵ, Arg @ (at[z, m] ϵ), .01]}];

twistArrow = {Dashed, Gray, Arrowheads @.015, Arrow[v @ twistArcCmplx]};


kArrow = {Black, ahSz, Arrow[v @ {w @ z, w @ z + k[z, m] Exp[I Arg @ (at[z, m] ϵi)]}]};

velArrow = {Orange, ahSz, Arrow[v @ {w @ z, w @ z + wVelocity[z, m]}]};


arrowsA = {ϵArrow, ϵiArrow};

arrowsB = {velArrow, atϵArrow, kArrow, atϵiArrow, ϵGrayArrow, twistArrow};

sqArr = Translate[Scale[ squigglyArrow, .5], v @ 1.2];


(* CURVES *)

uCircleCmplx = Table[Exp[I θ], {θ, Range[0, 2.02 Pi, .02]}];

(* check curvature on w curve: Table[If[k[z,m]< 0, k[z,m],""],{z,uCircleCmplx}]

   Min @ Table[ k[z,m],{z,uCircleCmplx}] *)

uCircle = {Dashed, Gray, Line[v @ uCircleCmplx]};


innerCircleCmplx[r_] := Table[ r Exp[I θ], {θ, Range[0, 2.01 Pi, .01]}];

innerCirclesCmplx = Table[innerCircleCmplx[r / 4], {r, Range[3]}];

innerCircles =
  Table[{hues[[i]], Line[v @ innerCirclesCmplx[[i]]]}, {i, Range[1, 3]}];


(* w circles *)
```

```
wInnerCirclesCmplx = Table[w @ z, {z, innerCirclesCmplx}];

wInnerCircles =
  Table[{hues[[i]], Line[v @ wInnerCirclesCmplx[[i]]]}, {i, Range[1, 3]}];


expCurveCmplx = Table[w @ z, {z, uCircleCmplx}];

expUnitCrv = {Gray, Dashed, Line[v @ expCurveCmplx]};


(* TEXT *)

default = 12;
large = 14;
small = 10;

data1 = {
    {"z", 1.2 z, default},
    {"1", 1.2, default},
    {"0", 0 - .1 - .1 I, {default, Bold}},
    {"e^(z^-2)", 2 - .5 I, default},
    {"(a)", -1.3 I, default},
    {"Figure 2.\nInterior to exterior", -2.5 I, default}
    (*{"(a)", -1.5I, default}*)
  };

data2 = {
    {"w", w @ z - .3, default},
    (* {"twist",
     .30 Exp[I (Arg @ (at[z,m] ϵ)-Arg @ ϵ)/2]+w @ z + 1-.4I,small}, *)
    {"κ̃ = " <> ToString[k[z, m] // N],
     w @ z + 1.2 k[z, m] Exp[I Arg @ (at[z, m] ϵi)], small},
    {"dw/dθ = " <> ToString[NumberForm[wVelocity[z, m] // N, 3]],
     w @ z + 1.2 wVelocity[z, m], small},
    {ToString[w @ 1], w @ 1 + .2, default},
    {"(b)", 1 - 4 I, default}
  };

textFig1 = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, data1}];

textFig2 = Table[Text[Style[item[[1]], item[[3]]], v @ item[[2]]], {item, data2}];


(* COLLECTED OBJECTS *)

figObjsA = {uCircle, arrowsA, innerCircles, zPoints, textFig1, sqArr};

figObjsB = {expUnitCrv, arrowsB, wInnerCircles, wPoints, textFig2};
```

```
GraphicsRow[
 {
  Graphics[{figObjsA}, Background → White,
   PlotRange → {{-1.50, 2.4}, {-4, 4}}, ImageSize → Medium],
  Graphics[{figObjsB}, Background → White,
   PlotRange → {{-4, 10}, {-5, 4}}, ImageSize → Medium]
 }, Spacings → -60
]
```