Needham, Chapter 5, Exercise 15, p. 259.
G. Palmer, January 29, 2016

15. (i) By noting the unit tangent (in the counterclockwise direction) to an origin centred circle can be written as $\xi = i(z/|z|)$, show that formula (23) for the curvature of the image of such a circle can be written as

$$\tilde{\kappa} = \frac{1+Re\left[\frac{zf''}{f'}\right]}{|zf'|}$$

$$\tilde{\kappa} = \frac{1}{|f'(p)|}\left(\mathrm{Im}\left[\frac{f''(p)\,\hat{\xi}}{f'(p)}\right] + \kappa\right) \qquad (23)$$

Substitute i(z/|z|) for ξ.
Substitute $\frac{1}{|z|}$ for k, because the equation for the circle can be written z =

r

e

Let f be understood to be f(z).

$$\tilde{\kappa} = \frac{1}{|f'|}\left(\mathrm{Im}\left[\frac{f''(iz/|z|)}{f'}\right] + \frac{1}{|z|}\right)$$

$$= \frac{1}{|zf'|}\left(\mathrm{Im}\left[\frac{if''z}{f'}\right] + 1\right)$$

$$= \frac{1+\mathrm{Re}\left[\frac{zf''}{f'}\right]}{|zf'|}$$

(ii) What should this formula yield if f(z) = log z? Check that it does.

log z sends circles to vertical lines. Since the circle is origin-centred, log z should send the circle to the y axis. Then $\kappa = 0$.

$$f' = (\log z)' = \frac{1}{z}$$

$$f'' =$$

$-z$

$$\tilde{\kappa} = \frac{1+\mathrm{Re}\left[\frac{z(-z^{-2})}{z^{-1}}\right]}{|z\,z^{-1}|}$$

$$= \frac{1+\mathrm{Re}[-1]}{1} = \frac{0}{1} = 0$$

as predicted.

(iii) What should this formula yield if f(z) = z^m? Check that it does. What is the significance of the value of $\tilde{\kappa}$ when m is negative? [Hint: Which way does the velocity complex number of the image rotate as z travels counterclockwise around the original circle?]

The absolute value of
z
$|z|$
$|z|$

f' = $(z^m)'$ =
mz

f'' =
$m(m-1)z$

$$\kappa = \frac{1+\text{Re}\left[\frac{z\,m(m-1)\,z^{m-2}}{\text{mz}^{m-1}}\right]}{\left|\text{zmz}^{m-1}\right|}$$

$$= \frac{1+\text{Re}\left[\frac{(m-1)\,z^{m-1}}{z^{m-1}}\right]}{\left|\text{mz}^{m}\right|}$$

$$= \frac{1+\text{Re}[m-1]}{\left|\text{mz}^{m}\right|} = \frac{1+m-1}{\left|\text{mz}^{m}\right|}$$

$$= \frac{m}{\left|\text{mz}^{m}\right|} = \frac{m}{|m||z^{m}|} =$$
sign(m)

$|z|$

The formula results
in

$|z|$

What is the significance of the value of $\tilde{\kappa}$ when m is negative? [Hint: Which way does the velocity complex number of the image rotate as z travels counterclockwise around the original circle?]

We consulted chapter 1, §II, 2 “Moving Particle Argument” (pp. 10-12) for the following. Let z =

re
z
r
e
d

(*r*

V =
im-
r

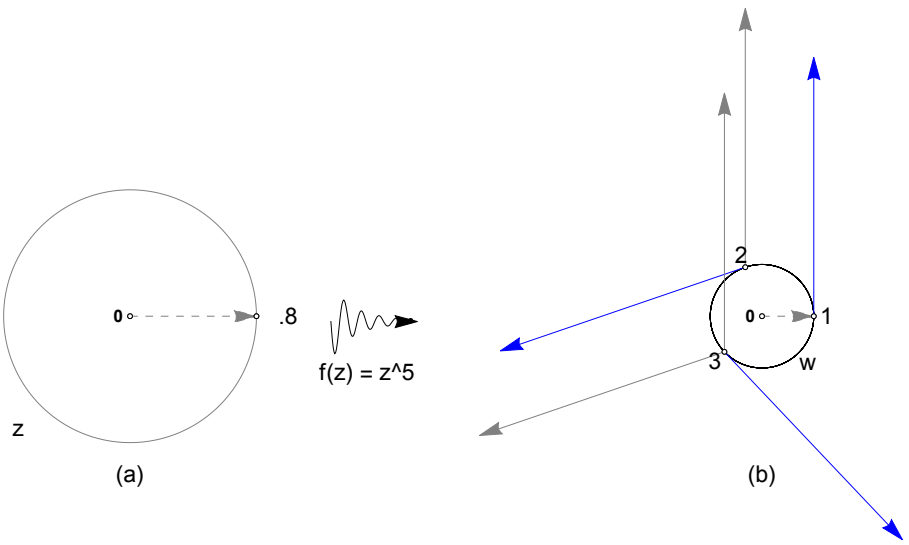
We see that when m > 0, V rotates mw by Pi/2 in the counterclockwise direction, so it is tangent to the image circle. As z travels in the counterclockwise direction, V must also rotate in the counterclockwise direction (Figure 1).

To illustrate the case when m < 0, let n = -m, i.e. n is a positive number, so V =
im
r
e
–in-
r

Figure 1 (a) shows a circle of radius .8 and a gray, dashed ray pointing to .8. This same arrow under f is shown in (b). Under f(z) =
z
r

r

r


0
.8
z
(a)
f(z) = z^5
2
0
1
3
w
(b)