Needham, Chapter 5, Exercise 14, pp. 259-260.
G. Palmer, June 15, 2016, 4:30 pm PST

14 (i) Look up both a stereographic map and a Mercator map in an atlas, and make sure you can relate the changes in shape you see to your understanding of the complex logarithm.

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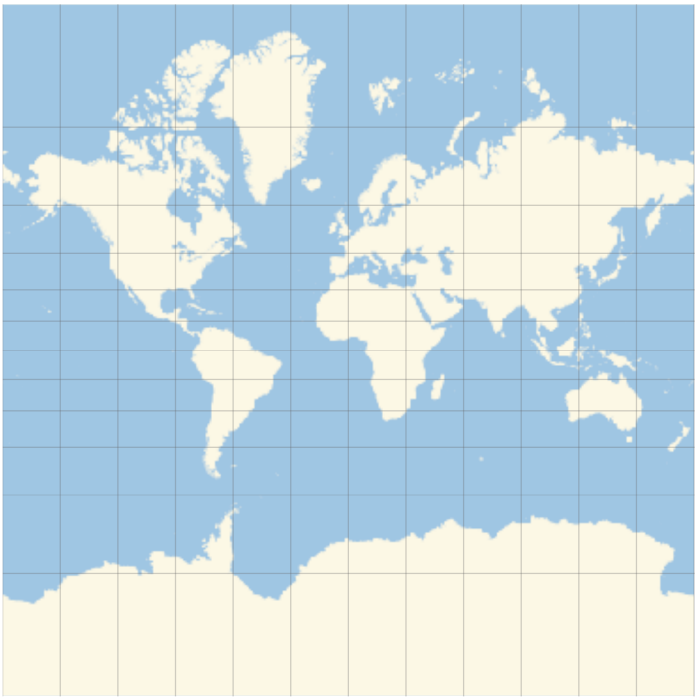


Figure 1. Mercator projection

The Mercator projection (Figure 1), plotted with *Mathematica*, appears to be expanded in the polar regions. We do not see this expansion in the polar projection, below left, nor in the equatorial stereographic projection, below right.* That is likely a result of the Log function, but we are not sure how it would be accomplished.

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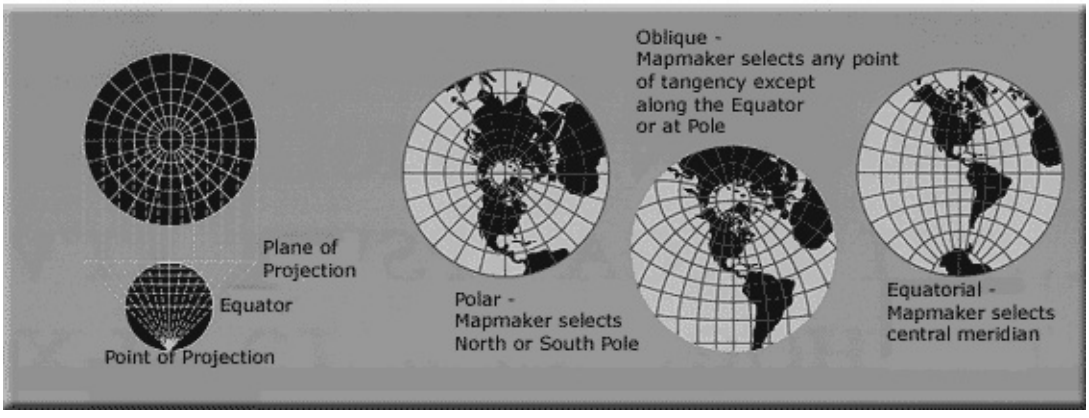


Figure 2. Stereographic projections

The lines of latitude on the Mercator projection are straight. Needham has shown that the complex logarithm is “uniquely defined (up to constants) as the conformal mapping sending concentric circles to parallel lines” (p. 222), so this might be used for a transformation of the polar stereographic projection (Figure 2, left) into the Mercator projection. We would lose most of the southern hemisphere, but that is a practical problem, not a logical one, as the southern hemisphere could be projected from the north pole and then transformed with Log. Our simulation will attempt to send the circles of a polar projection to a Mercator projection.

Figure 3 is a diagram of a unit circle on the ℂ plane seen from above, but the plane must be rotated on the x axis to be parallel to the z axis. This diagram was used to calculate the radius ℛ of the circles on the projection plane P = i, which touches the N pole and is orthogonal to the z axis.

Figure 4a presents a sphere with lines of latitude from just the northern hemisphere projected onto a plane P = i by stereographic projection (blue circles). In Figure 4b, a rectangular continent is plotted on P. Figure 5a presents a Mercator projection created by applying the Log function to the stereographic projection circles and the continent. Notice that the angles that the sides of the continent makes with the lines of latitude are preserved in the Mercator projection. Thus, on the Mercator, continents appear to shrink as one goes further from the pole. Figure 5b rotates the projection 90 degrees clockwise to present it from a more normal point of view with the equator (bottom line) running horizontally.

(ii) Imagine plotting a straight line course on a Mercator map and then actually traveling it on the high seas. Show that as you sail, the reading of your compass never changes.

In both Figures 5a and 5b, we can see that if we plot a straight line on the Mercator from the equator and follow that course northeast to a point on latitude 75, the reading of the compass should always point at roughly the same angle to the portside.

*<http://egsc.usgs.gov/isb//pubs/MapProjections/projections.html>

Out[103]=

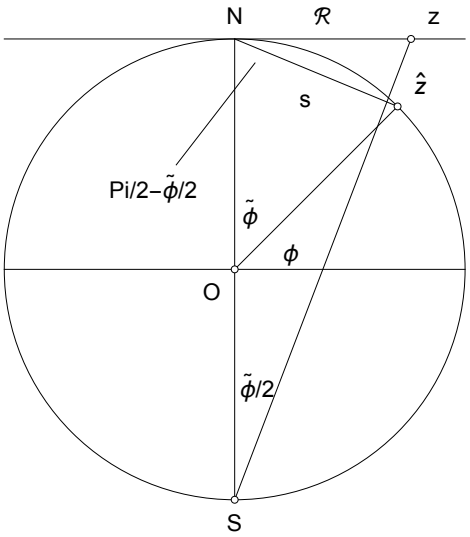


Figure 3. $\mathcal{R}$ = |2 Sin($\tilde{\phi}$/2)|

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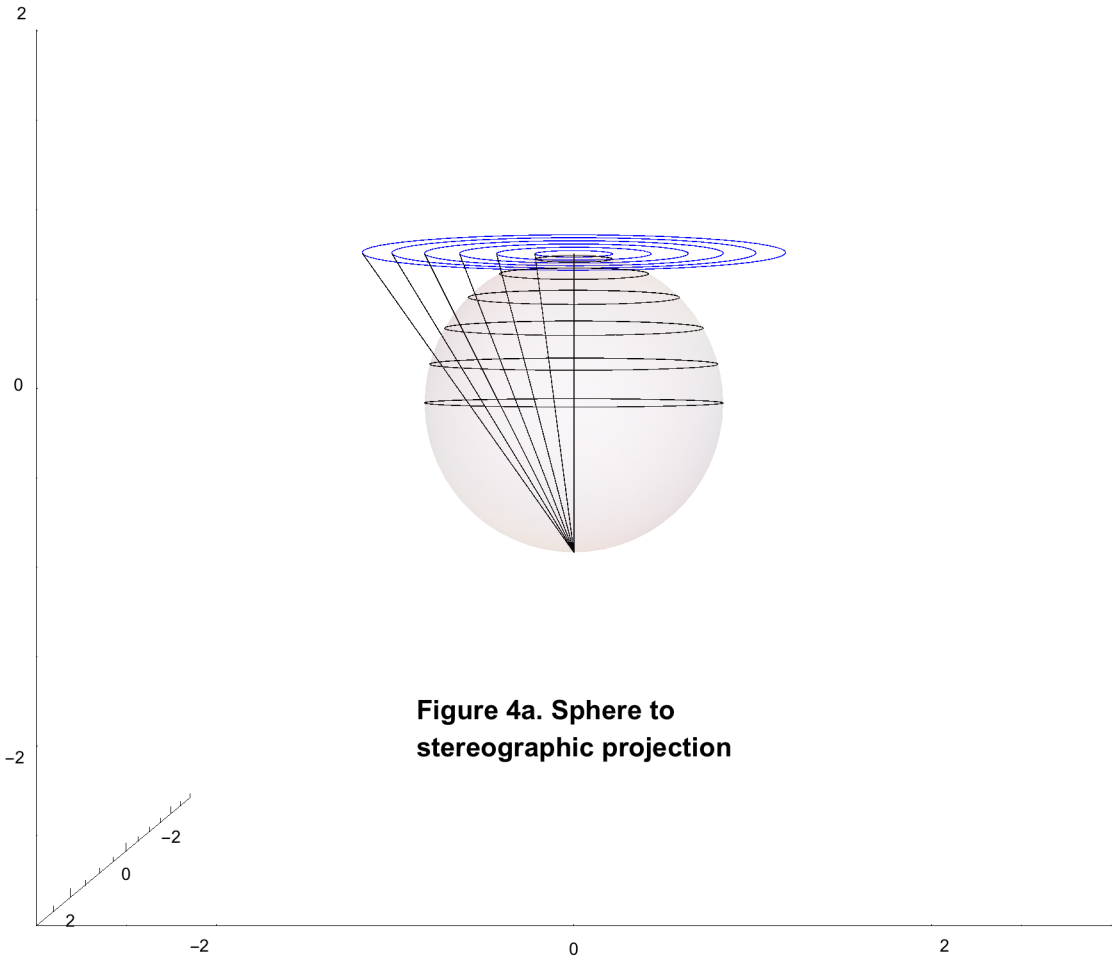


Figure 4a. Sphere to stereographic projection

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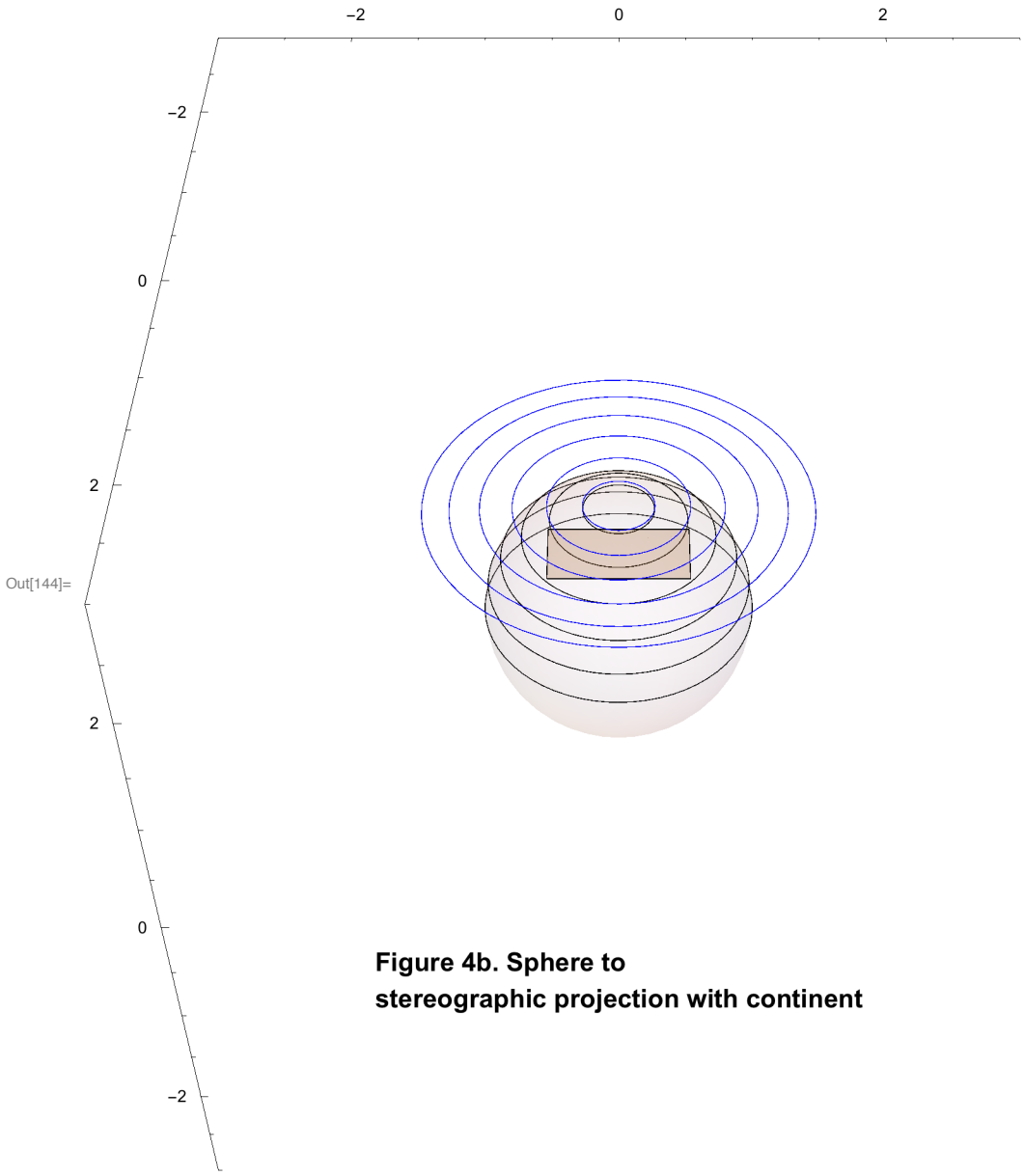


Figure 4b. Sphere to stereographic projection with continent

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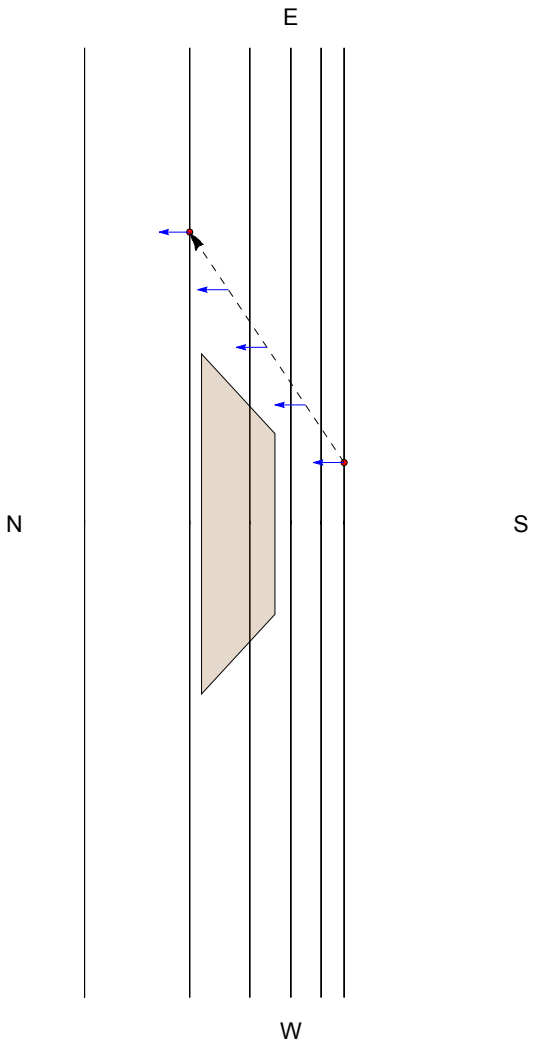


Figure 5a. Mercator map
Log[stereographic projection]
with ship's course (dashed arrow)

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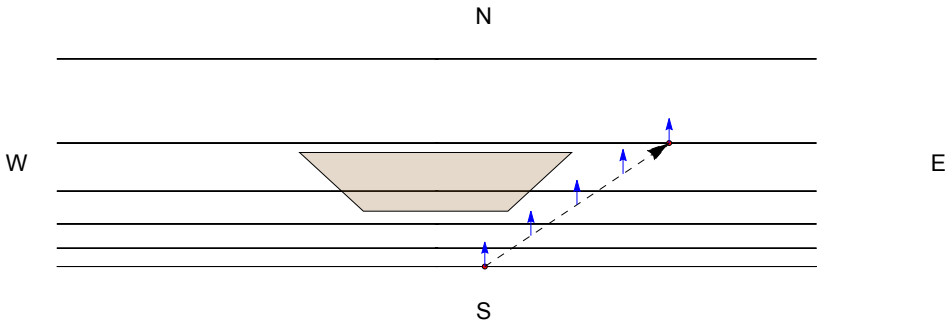


Figure 5b. Rotated Mercator map
with ship's course
and compass (blue)