Needham, Chapter 5, Exercise 13, p. 259.
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13. (i) If a is an arbitrary constant, show that the series

$$f(z) = 1 + az + \frac{a(a-1)}{2!} z^2 + \frac{a(a-1)(a-2)}{3!} z^3 + \ldots$$

converges inside the unit circle.

From p. 74, R = $\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$. Substitute the most general summand from the problem:

$$R = \lim_{n\to\infty} \left| \frac{\frac{a!}{(a-n)!\,n!}}{\frac{a!}{(a-(n+1))!\,(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a!\,(a-(n+1))!\,(n+1)!}{a!\,(a-n)!\,n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(a-(n+1))!\,(n+1)!}{(a-n)!\,n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{a-n} \right| = \left| \frac{\infty}{-\infty} \right| = 1$$

(because at ∞, 1 and a become infinitely small fractions of the numerator and denominator)

The radius of convergence is 1, so f(z) converges inside the unit circle.

(ii) Show that (1+z)f' = af

$$(1+z)f' = (1+z)\left(1 + az + \frac{a(a-1)}{2!} z^2 + \frac{a(a-1)(a-2)}{3!} z^3 + \ldots\right)'$$

$$= (1+z)\left(0 + a + \frac{2\,a(a-1)}{2!} z + \frac{3\,a(a-1)(a-2)}{3!} z^2 + \ldots\right)$$

$$= (1 + z)\left(a + \frac{a(a-1)}{1!} z + \frac{a(a-1)(a-2)}{2!} z^2 + \frac{a(a-1)(a-2)(a-3)}{3!} z^3 \ldots\right)$$

$$= \left(a + \frac{a(a-1)\,z}{1!} + \frac{a(a-1)(a-2)}{2!} z^2 + \frac{a(a-1)(a-2)(a-3)}{3!} z^3 + \ldots\right) +$$

$$\left(az + \frac{a(a-1)}{1!} z^2 + \frac{a(a-1)(a-2)}{2!} z^3 + \ldots\right)$$

$$= a + \left(a(a-1) + a\right) z + \left(\frac{a(a-1)(a-2)}{2!} + \frac{a(a-1)}{1!}\right) z^2 + \left(\frac{a(a-1)(a-2)(a-3)}{3!} + \frac{a(a-1)(a-2)}{2!}\right) z^3 + \ldots)$$

$$= a + a^2 z + a\left(\frac{(a-1)(a-2)}{2!} + \frac{2(a-1)}{1!\ \ 2}\right) z^2 + a\left(\frac{(a-1)(a-2)(a-3)}{3!} + \frac{3(a-1)(a-2)}{2!\ \ 3}\right) z^3 + \ldots$$

$$= a(1 + az + \frac{(a-2+2)(a-1)}{2!}z^2 + \frac{(a-1)(a-2)(a-3)+3(a-1)(a-2)}{3!}z^3 + \ldots)$$

$$= a(1 + az + \frac{a(a-1)}{2!}z^2 + \frac{(a-3+3)(a-1)(a-2)}{3!}z^3 + \ldots)$$

$$= a(1 + az + \frac{a(a-1)}{2!}z^2 + \frac{a(a-1)(a-2)}{3!}z^3 + \ldots))$$

$$= af$$

(iii) Deduce that $[(1 + z)^{-a} f]' = 0$.

$$[(1 + z)^{-a} f]' = ((1 + z)^{-a})' f + (1 + z)^{-a} f'$$

$$= -a(1 + z)^{-a-1} f + (1 + z)^{-a} \frac{af}{1+z}$$

$$= -a(1 + z)^{-a-1} f + a(1 + z)^{-a-1} f$$

$$= 0$$

(iv) Conclude that $f(z) = (1 + z)^a$.

"f" is understood to mean "f(z)".

$[(1 + z)^{-a} f]' = 0$, from (iii)

$(1 + z)^{-a} f = c$ (c a constant by integration).

$f(z) = (1 + az + \frac{a(a-1)}{2!} z^2 + \frac{a(a-1)(a-2)}{3!} z^3 + \ldots)$, from (i)

When z = 0, f = 1, so

$(1 + z)^{-a} f = c$

$1 \cdot 1 = c$

$c = 1$

Then

$(1 + z)^{-a} f = 1$

$f(z) = (1 + z)^a$