Needham, Chapter 5, Exercise 12, p. 259.
G. Palmer, January 18, 2016

12. Provided it is properly interpreted, show that $(z^{\mu})' = \mu z^{\mu-1}$ *is still true even if* μ *is complex.*

The qualifying phrase “provided it is properly interpreted” suggests that z^{μ} might be a multifunction with many values. We can rewrite $z^{\mu} = e^{\mu \log z}$. We let μ = a + ib. If b = 0, then μ = a and $z^{\mu} = e^{a \log z}$. We examine the possiblities in case b = 0.

If μ is a real number or integer

$$(z^a)' = (e^{a \log z})'$$

$$= e^{a \log z}(a \log z)'$$

$$= e^{a \log z}\left(\frac{a}{z}\right)$$

$$= a\, z^a/z$$

$$= a\, z^{a-1}$$

This satisfies the equation $(z^{\mu})' = \mu z^{\mu-1}$.

If a = p/q is a rational fraction, then z^{μ} is a multifunction with many values (pp. 101-102, 230), as seen by the first term of the following result, which contains the integer n in the numerator of the power. Square brackets signify the principal branch of the log function, as does the capital L on Log. It is given, where $z = re^{i\theta}$, that Log z = ln r + iθ and log z = Log z + i2nPi.

$$z^{p/q} = e^{(p/q) \log z}$$

$$= e^{(p/q)\,(\mathrm{Log}\, z + \mathrm{i2nPi})}$$

$$= e^{(p/q)\,\mathrm{i2nPi}}\, e^{(p/q)\,\mathrm{Log}\, z}$$

$$= e^{\frac{\mathrm{i2npPi}}{3}}\left[z^{p/q}\right] \qquad (1)$$

Then $(z^{\mu})' = \left(e^{\frac{\mathrm{i2npPi}}{3}}\left[z^{p/q}\right]\right)'$

$$= e^{\frac{\mathrm{i2npPi}}{3}}\left[z^{p/q}\right]'$$

$$= e^{\frac{\mathrm{i2npPi}}{3}}(\mathrm{p/q})\left[z^{(p/q\,-\,1)}\right]$$

$$= e^{\frac{\mathrm{i2npPi}}{3}}\frac{(p/q)\left[z^{p/q}\right]}{z} \qquad (2)$$

Needham (p. 230) says, where “a” is a fraction,

The amplitwist of each branch of z^a is given by $(z^a)' = a\, z^a / z$, *provided that the same branch of z^a is used on both sides of the equation.* We substitute (1) into the LHS and (2) into the RHS:

$$(e^{\frac{i2npPi}{3}}[z^{p/q}])' = e^{\frac{i2npPi}{3}}\frac{(p/q)[z^{p/q}]}{z}$$

$$e^{\frac{i2npPi}{3}}[z^{p/q}]' = e^{\frac{i2npPi}{3}}\frac{(p/q)[z^{p/q}]}{z}$$

$[z^{p/q}]' = \frac{(p/q)[z^{p/q}]}{z}$, by dividing both sides by the branch factor $e^{\frac{i2npPi}{3}}$

Substituting a for p/q and z^a for $[z^{p/q}]$, gives (a z^a / z), so our calculation bears out the equation for amplitwist *provided that the same branch of z^a is used on both sides of the equation*. The equation for amplitwist of a fractional exponent of a complex number satisfies the equation $(z^\mu)' = \mu\, z^{\mu-1}$.

In the case that μ is complex, let μ = a + ib. Then

$$z^\mu = e^{\mu \log z}$$

$$= e^{\mu\,(\ln r + i(\theta + 2\,\mathrm{nPi}))}$$

$$= e^{\mu i 2\mathrm{nPi}} e^{\mu\,\mathrm{Log}\,z}$$

$$= e^{i\mu 2\mathrm{nPi}}[z^\mu]$$

$$(z^\mu)' = \left(e^{i\mu 2\mathrm{nPi}} e^{\mu\,\mathrm{Log}\,z}\right)'$$

$$= e^{i\mu 2\mathrm{nPi}}\left(e^{\mu\,\mathrm{Log}\,z}\right)'$$

$$= e^{i\mu 2\mathrm{nPi}} e^{\mu\,\mathrm{Log}\,z}(\mu\ \mathrm{Log\ z})'$$

$$= e^{i\mu 2\mathrm{nPi}}[z^\mu]\mu\ (\mathrm{Log\ z})'$$

$$= e^{i\mu 2\mathrm{nPi}}\,\mu[z^\mu](1/z)$$

Then

$$\left(e^{i\mu 2nPi}[z^\mu]\right)' = e^{i\mu 2\mathrm{nPi}}\,\mu[z^\mu]/z$$

Divide by the branch term on both sides

$$[z^\mu]' = \mu[z^\mu]/z = \mu\left[z^{\mu-1}\right]$$

Again, provided it is properly interpreted so that the same branch factors are used on both sides of the equation, we can write $(z^{\mu})' = \mu z^{\mu-1}$.

As a point of interest, the branch factor $e^{i\mu 2nPi}$ expands to $e^{-2\,\mathrm{nPi}(b-\mathrm{ia})}$.