Needham, Chapter 5, Exercise 11, p. 259.
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11. Use $(e^z)' = e^z$ *to show that the derivatives of all the trig functions are given by the familiar rules of real analysis.*

sin, cos, tan, cotan, sec (1/cos), cosec (1/sin)

$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

$\tan z = \frac{e^{iz}-e^{-iz}}{2i} / \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$

$\text{cotan } z = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$

$\sec z = \frac{2}{e^{iz}+e^{-iz}} = 1/\cos z, \qquad \csc z = \frac{2i}{e^{iz}-e^{-iz}} = 1/\sin z$

It will be useful to rearrange some of these and take squares and derivatives:

$(e^{iz} + e^{-iz})^2 = 4/\sec^2 z = 4\cos^2 z$

$(e^{iz} - e^{-iz}) = 2i \sin z$

$(e^{iz} + e^{-iz}) = 2 \cos z$

$(e^{iz} + e^{-iz})^2 = 4 \cos^2 z$

$(e^{iz} - e^{-iz})^2 = -4 \sin^2 z$

$(e^{iz} - e^{-iz})' = i(e^{iz} + e^{-iz})$

$(e^{iz} + e^{-iz})' = i(e^{iz} - e^{-iz})$

(sin z)' $= \frac{1}{2i}((e^{iz})' - (e^{-iz})')$

$= \frac{1}{2i} i (e^{iz} - (-e^{-iz}))$ [because $(e^{az})' = ae^{az}$]

$= \frac{1}{2}(e^{iz} + e^{-iz})$

$= \cos z$

(cos z)' $= \frac{1}{2}(e^{iz} + e^{-iz})'$

$$= \frac{i}{2}\left(e^{iz} - e^{-iz}\right)$$

$$= \frac{-1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= -\sin z$$

(sec z)' = $(\frac{2}{e^{iz}+e^{-iz}})'$ = (1/cos z)'

By the quotient rule, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

Let g = $e^{iz} + e^{-iz}$, then g' = $i\left(e^{iz} - e^{-iz}\right)$

$$(\sec z)' = \left(\frac{2}{g}\right)' = \frac{-2g'}{g^2}$$

$$= \frac{-2i\left(e^{iz} - e^{-iz}\right)}{\left(e^{iz}+e^{-iz}\right)^2}$$

$$= \frac{-2i\,(2i\sin z)}{4\cos^2 z}$$

$$= \frac{\sin z}{\cos^2 z} = \sec z \tan z$$

(csc z)' = $(\frac{2i}{e^{iz}-e^{-iz}})'$ = (1/sin z)'

Let f = 2i, g = $e^{iz} - e^{-iz}$

$$= \frac{-2i\left(e^{iz}-e^{-iz}\right)'}{\left(e^{iz}-e^{-iz}\right)^2}$$

$$= \frac{-2i^2\left(e^{iz}+e^{-iz}\right)}{-4\sin^2 z}$$

$$= \frac{2\,(2\cos z)}{4\sin^2 z}$$

$$= -\frac{\cos z}{\sin^2 z} = -\csc z \cot z$$

(tan z)' = $\left(\frac{e^{iz}-e^{-iz}}{i\left(e^{iz}+e^{-iz}\right)}\right)'$

We again use the quotient rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$= \frac{\left(e^{iz}-e^{-iz}\right)' i\left(e^{iz}+e^{-iz}\right)-\left(e^{iz}-e^{-iz}\right)\left(i\left(e^{iz}+e^{-iz}\right)\right)'}{\left(i\left(e^{iz}+e^{-iz}\right)\right)^2}$$

$$= \frac{i\left(e^{iz}+e^{-iz}\right) i\left(e^{iz}+e^{-iz}\right) -\left(e^{iz}-e^{-iz}\right)\left(-\left(e^{iz}-e^{-iz}\right)\right)}{-\left(e^{iz}+e^{-iz}\right)^2}$$

$$= 1 - \frac{-4\sin^2 z}{4\cos^2 z}$$

$$= 1 + \tan^2 z = \sec^2 z$$

(cot z)' $= \left(\frac{i\left(e^{iz}+e^{-iz}\right)}{e^{iz}-e^{-iz}}\right)'$

$$= \frac{-\left(e^{iz}-e^{-iz}\right)\left(e^{iz}-e^{-iz}\right)-i\left(e^{iz}+e^{-iz}\right)\left(e^{iz}-e^{-iz}\right)'}{\left(e^{iz}-e^{-iz}\right)^2}$$

$$= \frac{-i\left(e^{iz}-e^{-iz}\right)^2 - i\left(e^{iz}+e^{-iz}\right) i\left(e^{iz}+e^{-iz}\right)}{\left(e^{iz}-e^{-iz}\right)^2}$$

$$= -1 + \frac{\left(e^{iz}+e^{-iz}\right)^2}{\left(e^{iz}-e^{-iz}\right)^2}$$

$$= \frac{4\cos^2 z}{-4\sin^2 z} - 1$$

$$= -\cot^2 z - 1$$

$$= -(1 + \cot^2 z)$$