Needham, Chapter 5, Exercise 1, p. 235
G. Palmer, December 17, 2015

1. Show that if f = u + iv is analytic, then $\nabla_u \bullet \nabla_v = 0$, where ∇ is the gradient operator of vector calculus. Explain this geometrically.

If f = u + iv is analytic, then by the Cauchy-Riemann equations,

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_x v = -\partial_y u$$

From vector calculus, we get the gradients for u(x,y) and v(x,y).

$\nabla_u = (\partial_x u, \partial_y u)$ from vector calculus

$\nabla_v = (\partial_x v, \partial_y v)$ "

Then

$$\nabla_u \bullet \nabla_v = (\partial_x u, \partial_y u) \bullet (\partial_x v, \partial_y v)$$

$$= \partial_x u \, \partial_x v + \partial_y u \, \partial_y v$$

$$= \partial_y v \, \partial_x v - \partial_x v \, \partial_y v \quad \text{by Cauchy-Riemann substitution}$$

$$= 0$$

Explain this geometrically.

f = u + iv takes x and y to u and v. ∇_u is the change in u with respect to infinitesimal changes in x and y. ∇_v is the change in v with respect to infinitesimal changes in x and y. We can see this is so, because f is analytic, so infinitestimal vectors under f have amplitwist. The amplitwist takes $\vec{\epsilon}$ and $i\vec{\epsilon}$ to $\overrightarrow{\epsilon u}$ and $i\overrightarrow{\epsilon v}$. Since $\vec{\epsilon}$ on the x axis and $i\vec{\epsilon}$ on the y axis (p. 216, Figure 1) are orthogonal, ∇_u, which is on the u axis, must be orthogonal to ∇_v, which is on the v axis. Hence, $\nabla_u \bullet \nabla_v = 0$. Furthermore, since f has amplitude and $|i\vec{\epsilon}| = |\vec{\epsilon}|$, the magnitudes $|\overrightarrow{\epsilon u}|$ and $|\overrightarrow{\epsilon v}|$ are equal.

Vasco used the idea of a complex vector, which is simply a complex number. ∇_u and ∇_v are complex vectors. I reproduce it here from memory.

$$\nabla_u = \partial_x u + i\partial_y u$$

$$\nabla_v = \partial_x v + i\,\partial_y v$$

Now, if we can show that $i\nabla_v = \nabla_u$ or $-\nabla_u$ without violating the Cauchy-Riemann equations, then it follows that $\nabla_u \bullet \nabla_v = 0$. If ∇_u is orthogonal to ∇_v, then so is $-\nabla_u$.

$$i\nabla_v = i(\partial_x v + i\partial_y v)$$

$= i\,\partial_x v \; - \partial_y v$

$= -i\,\partial_y u - \partial_x u$ by Cauchy-Riemann substitution

$= -\nabla_u$

This is what we wanted.

Note on Figure 1: In the figure f = z^2. To draw the small blue arrows on the RHS, I resorted to amplitwist as shown by the labels. The u and v coordinates and the partials of u and v were plotted in green with the same twist as $\vec{\epsilon}$ and i$\vec{\epsilon}$. I'm not sure that is valid, so if and when I get a better understanding of how to determine the difference in coordinate orientation between the z plane and the f(z) plane, I will fix it. I think this is what one would expect the plot to look like, but it doesn't fall out naturally (for me) from the theory. The length of the arrows for the partials is arbitrarily set at 1/4.

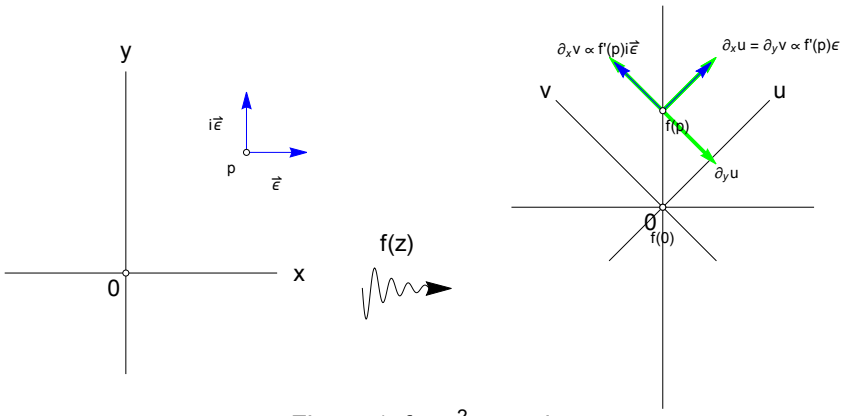


Figure 1. f = z^2 = u + iv