Needham, Chapter 4, Exercise 8, p. 212.
G. Palmer, November 2, 2015

8. Use pictures to explain why if f(z) is analytic on some connected region, each of the following conditions forces it to reduce to a constant.

(i) Re f(z) = 0

$f(z) = u + v\,i$

Since u = (Re f(z)) = 0, f(z) = v i, which is a way of writing the curve is the y axis.

Figure 1 shows that the infinitesimal vectors (blue and green) go to the point f(z) = vi on the i axis. They are drawn with larger overlapping points at f(z) to reveal their location. Since they go to f(z) and emanate from f(z), they must align with the i axis, so they rotate by different amounts and have no twist. It hardly matters, because their length at f(z) is zero, so amplification is zero. With zero amplitude, $f'(z) \equiv 0$ on a connected region. From 7(ii) we know that if $f'(z) \equiv 0$ on some connected region, f(z) is a constant there. Having zero amplitude is enough to qualify f as analytic (Vasco).

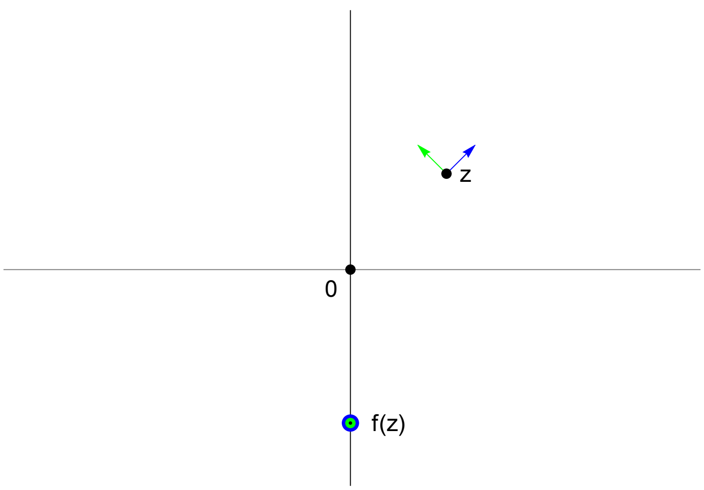

Figure 1. Re F(z) = 0

(ii) |f(z)| = const

|f(z)| = const describes a circle and f(z) lies on the circle for all z. Hence, under f(z) all infinitesimal vectors in the neighborhood of z go to the circle as tangents. All vectors emanating from z are therefore rotated by different amounts at f(z). Hence there is no twist and f can only be analytic if amplification equals zero, in which case $f'(z) \equiv 0$. From 7(ii) we know that if $f'(z) \equiv 0$ on some connected region, f(z) is a constant there. We also deduce that the only point on the “circle” is the single point f(z). Since the infinitesimal vectors also go to f(z), they must have zero length. We are a little inconsistent, because these zero vectors could also be represented as points as in Figure 1.

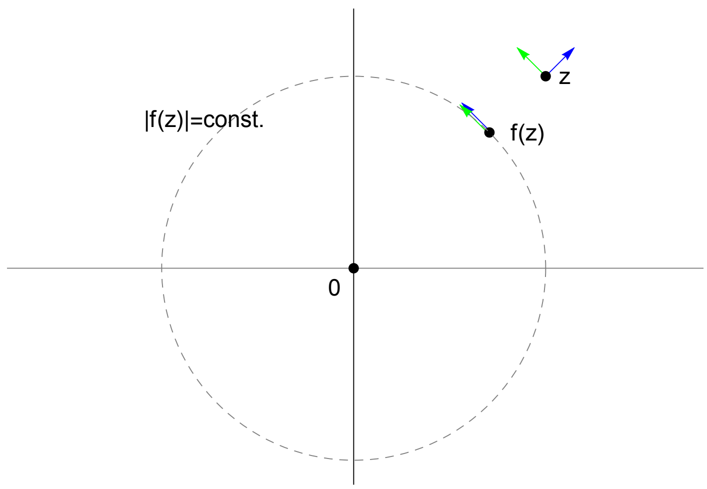

Figure 2. |f(z)| = const.

(iii) Not only is f(z) analytic, but $\overline{f(z)}$ is, too.

~~Figure 3. f(z) and $\overline{f(z)}$ analytic implies that f(z) = $\overline{f(z)}$ equals a real number. Otherwise, only one of them could be analytic by p. 203: "Thus if f(z) is a mapping that is known to possess an amplification throughout an infinitesimal neighbourhood of a point p, then either f(z) is analytic at p, or else conj(f(z)) is analytic at p." Figure 3 shows that both the infinitesimal vectors go to a single point f(z) on the real line. Since infinitesimal vectors also go to the real line, they have different changes of direction and no twist. Since f(z) and $\overline{f(z)}$ are analytic, the amplification is zero, meaning that f′(z) = $\overline{f(z)}'$ ≡ 0, so f(z) = $\overline{f(z)}$ = constant.~~

That f(z) and $\overline{f(z)}$ are analytic on a connected region implies that both have amplification but not twist, for otherwise only one or the other would be analytic (p. 203). If the ϵ arrows emanating from z are carried to f(z), they are flipped by $\overline{f(z)}$ and their sense is reversed. Hence, they lack twist. Since they have amplification, but not twist, the amplification must be zero. Since the amplification is zero, f′(z) ≡ 0, because f′(z) is equivalent to A = $ae^{i\theta}$, where a is amplification (p. 199). If f′(z) ≡ 0, f(z) is a constant by Exercise 7.

Comment: Logically, since f(z) = const, $f(\epsilon_0)$ = const, so the length of the ϵ_1 arrows emanating from f(z) and $\overline{f(z)}$ is zero. Hence, there can be no flip, no difference in length of the arrows, and no twist. This is rather like the situation in (ii).

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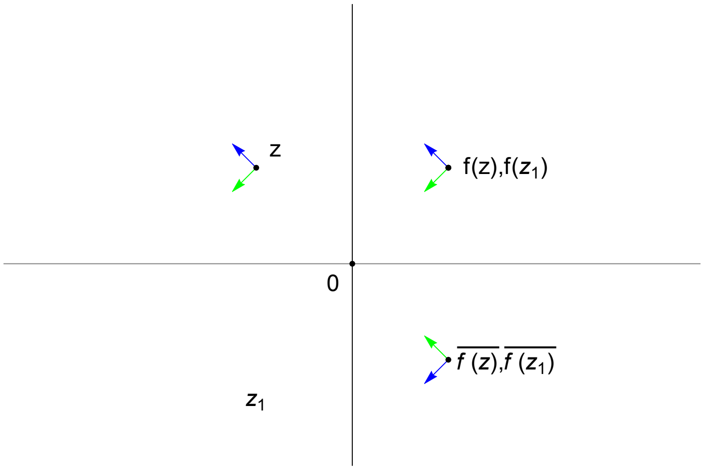

Figure 3. Both f(z) and $\overline{f\,(z)}$ analytic
f(z) = const