Needham, Chapter 4, Ex. 6, p. 211.
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6. Continuing from the previous exercise, show that no choice of v can make f(x+iy) = $\left(x^2 + y^2\right)$ + iv analytic.

We make the assumption that x and y are real. Rewrite f(x+iy) = $\left(x^2 + y^2\right) + \text{iv}$ as

f(z) = u + iv, where u = $\left(x^2 + y^2\right)$ is a real number.

Then

$$\frac{\partial}{\partial x}(u) = 0$$

$$\frac{\partial}{\partial y}(\mathrm{u}) = 0$$

f can not be analytic because it has zero partials.

We can show it another way, also assuming that x and y are real.

$$\frac{\partial}{\partial x}(u) = 2x$$

$$\frac{\partial}{\partial y}(\mathrm{u}) = 2y$$

By Cauchy-Riemann

$$2x = \frac{\partial}{\partial y}(\mathrm{v})$$

$$\frac{\partial}{\partial x}(v) = -2y$$

Integrating

$$v = \int \frac{\partial}{\partial y}(v)\, dy = 2xy + c1$$

$$v = \int \frac{\partial}{\partial x}(v)\, dx = -2xy + c2$$

$$2v = 2xy - 2xy + c1 + c2$$

$$v = c$$

$$f(x+iy) = \left(x^2 + y^2\right) + iv = \left(x^2 + y^2\right) + ic$$

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial y}(v) \Rightarrow 2x = 0$$

$$\frac{\partial}{\partial x}(v) = -\frac{\partial}{\partial y}(u) \Rightarrow 0 = 2y$$

See Vasco's answer for a proof that doesn't assume that x and y are real numbers.

7. **(i) If g(z) = 3+2i then explain geometrically why g'(z) ≡ 0.**

3+2i is a fixed point. Amplication and rotation of z are different for every z. Therefore, g can have neither twist nor amplification and g'(z) ≡ 0.

(ii) Show that if the amplification of an analytic function is identically zero (i.e., f'(z) ≡ 0) on some connected region, then the function is constant there.

We begin with the idea that there is an infinitesimal vector that takes its direction and amplification from the curve on which it is drawn (Figure [3], p. 191. To illustrate the effect of f on a local region, two or more infinitesimal vectors may be drawn emanating from a point on the curve, which also happens to be located at the end of a vector drawn from the origin to f(z) on the curve. **One can not determine the amplitwist without considering the curve**, which in cases having amplitwist greater than zero, involves at least two points, e.g. $f(z_0)$ and $f(z_1)$. If the amplification is zero at all points in a conformal region, the twist is also zero (p. 201, part 2 Conformality Throughout a Region), and infinitesimal vectors at z map to 0 at f(z). If f(z) is analytic, it follows that there has been no change in f(z). It is constant.

Since an analytic function is also conformal, we can apply the principle to this problem involving a connected region. Vasco's answer illustrates a situation in which points within an infinitesimal connected disc are all mapped to the same point by an analytic function f(z) with zero amplification and twist. Since amplitwist is zero, there can have been no change in f(z). "f(z) is constant on the disc" (Vasco). z can be shifted throughout the connected region so that each new z has a connected disc that overlaps with the previous. All points within these discs map to f(z) with zero amplification and zero twist. Hence, f is constant on the connected region.

(iii) Give a simple counter example to show that this conclusion does not follow if the region is instead made up of disconnected components.

Consider the region R consisting of the upper half-plane (including the real line) plus the lower half plane, which we take to be disconnected because each has its own automorphism.

R = UHP + LHP

The proposition to be disproven: If the amplification of a function is identically zero on R, then it is constant on the UH-P and the LH-P. Suppose f(z) = i for all z in the upper half plane and f(z) = -i for all z in the lower half plane. Then f'(z) ≡ 0 on the disconnected region R, but f(z) is not constant on R.