

5. Consider $f(x+iy) = (x^2 + y^2) + i(y/x)$. Find and sketch the curves that are mapped by f into (a) horizontal lines, and (b) vertical lines. Notice from your answers that f appears to be conformal. Show that it is not in two ways: (1) by explicitly finding some curves whose angle of intersection isn't preserved; and (ii) by using the Cauchy-Riemann equations.

Rewrite

$$f(x+iy) = f(u,v) = u + iv, \text{ where } u = x^2 + y^2 + i(y/x) \text{ and } v = y/x$$

(a) The curves mapped into horizontal lines are curves with derivative 0, so the ratio of y and x must remain constant, as in straight lines.. They are straight lines passing through the origin as rays in Figure 1.

$$v = y/x = a$$

$$f(u,v) = u + a i \quad (\text{horizontal lines})$$

(b) The curves mapped into vertical lines are circles. The value of $x^2 + y^2$ remains constant, but the value of y/x varies.

$$u = x^2 + y^2 = b$$

$$f(u,v) = b + v i$$

To see this more clearly, we could write

$$f(z) = |z|^2 + i \tan z$$

showing for (a) that the imaginary term $\tan z$ is constant and the real term varies, while for b the tangent varies and the real term $|z|^2 = b$ is the equation for a circle.

Since conformal transformations map circles and lines to circles or lines, f appears to be conformal. Conformal transformations also preserve angles. The orthogonal angles at the intersections of the circles and rays appear to be preserved in the intersections of the horizontal and vertical lines.

Figure 2 shows hyperbolas and ellipses on the left hand side. These were generated by applying $\cos(z)$ to evenly spaced vertical and horizontal lines respectively, as on p. 89, Figure [27]. The right hand side shows the effect of $f(x+iy)$, which has sent the hyperbolas to ellipses and the ellipses to bell-shaped curves on the y axis and to vertical lines. The orthogonal angles of intersection of the ellipses and hyperbolas on the LHS are not preserved.

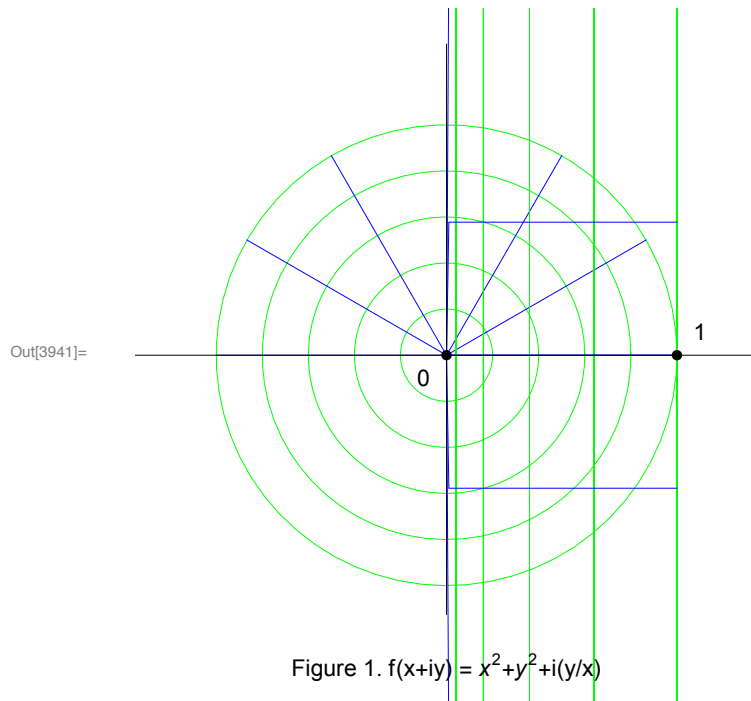
Cauchy-Riemann

$$f(x+iy) = (x^2 + y^2) + i(y/x)$$

$$u = (x^2 + y^2), \quad v = y/x$$

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial y}(v), \quad 2x \neq 1/x \quad \text{not satisfied}$$

$$\frac{\partial}{\partial x}(v) = -\frac{\partial}{\partial y}(u), \quad -x^{-2}y \neq -2y \quad \text{not satisfied}$$



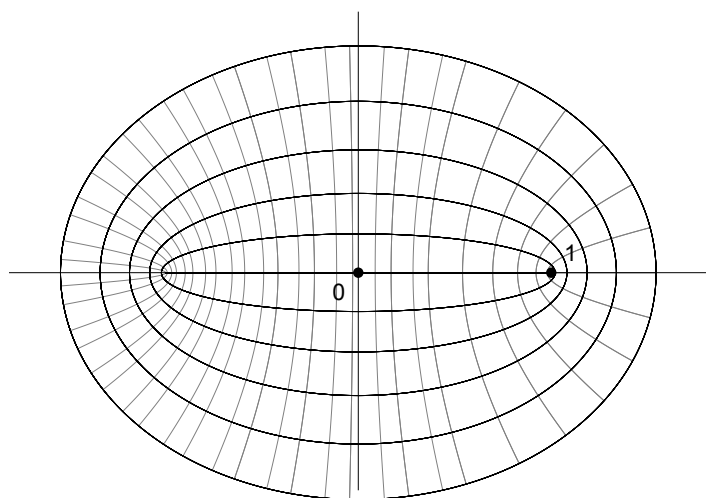


Figure 2. $f(x+iy) = x^2 + y^2 + i(y/x)$
not conformal $\rightarrow$

