Needham, Chapter 4, Exercise 21, p. 214-215.
G. Palmer, November 30, 2015

21. (i) By extending the argument given in the text, show that in three-dimensional space the effect of a linear transformation is to stretch space in three mutually perpendicular directions (generally by three different factors), then to rotate it.

We follow the argument in section 2, The Geometry of Linear Transformations, p. 208-209. Add a z coordinate for the third dimension. We can now draw a plane $\mathbb{C}_{yz}$ on the y and z axis (Figure 1). This plane is perpendicular to $\mathbb{C}$ and shares the y axis with $\mathbb{C}$.

Consider the effect of a linear transformation on a circle C_{xy} in the $\mathbb{C}_{yz}$ plane. As on the $\mathbb{C}$ plane, parallel lines map to image parallel lines and midpoints of line segments map to midpoints of image line segments.

Now consider the effect of a linear transformation on an infinitestimal sphere σ. The linear change of coordinates will lead to another quadratic equation on both the $\mathbb{C}$ and $\mathbb{C}_{yz}$ planes. The surface of intersection of these two new quadratic equations is an ellipsoid. Local transformations on the ellipsoid stretch the ellipsoid in three directions: d on the long axis of $\mathbb{C}$, e on the short axis of $\mathbb{C}$, and f on either the long or the short axis of $\mathbb{C}_{yz}$, as the two planes share the y axis (Figure 1).

The transformation requires six bits of information: (1) d, a direction of stretch; (2-4) amplification along directions d, e, and f; and (5,6) rotation on the $\mathbb{C}$ and $\mathbb{C}_{yz}$ planes. Directions e and f are predetermined as orthogonal to d and to one another. This appears to show that in three-dimensional space the effect of a linear transformation is to stretch space in three mutually perpendicular directions (generally by three different factors), then to rotate it in either or both of two planes.

Figure 1 shows a sphere that is expanded by different amounts on each dimension to an ellipsoid and rotated and in both planes.

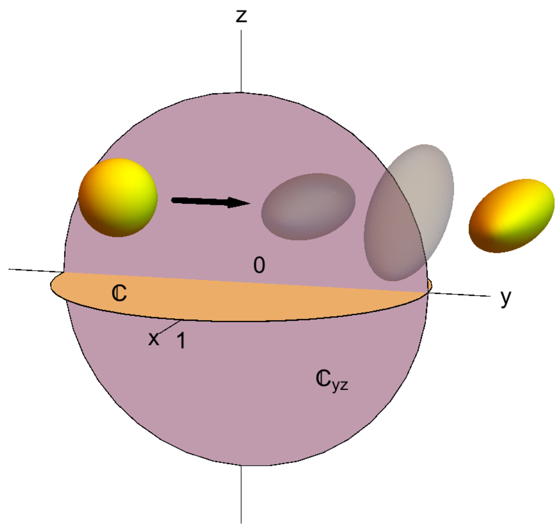

Figure 1. Transformation of sphere in 3 dimensions

(ii) Deduce that a mapping of three-dimensional space is locally a three-dimensional amplitwist if and only if it maps infinitesimal spheres to infinitesimal spheres.

In order for the quadratic equations on both planes (and the equation of intersection) to be analytic, the amplifications along d, e, and f must be equal and there must be twists on $\mathbb{C}$ and $\mathbb{C}_{yz}$. If this is the case, ϵ spheres are mapped to ϵ spheres. This reduces the degrees of freedom to 3: amplification, twist on $\mathbb{C}$ and twist on $\mathbb{C}_{yz}$. This is a three dimensional amplitwist. Since expansion is the same in all directions, the direction d of expansion is irrelevant. This three dimensional linear transformation with amplitwist appears to have only one additional degree of freedom compared to the two dimensional conformal transformation.

If the mapping were to send, say, ϵ spheres to non-spherical ϵ ellipsoids, as in Figure 1, or ϵ boomerangs, it would be obvious that amplification was not preserved. If it were to send ϵ spheres to ϵ mushrooms or pie slices, it would indicate that twist was not preserved. Only spheres have the uniformity in three dimensions to attest to the preservation of both twist and amplitude.

Now (perhaps) we have the following:

An orientation preserving mapping in three dimensions is conformal if and only if it sends infinitesimal spheres to infinitesimal spheres.

(iii) Deduce that inversion in a sphere preserves the magnitude of the angle contained by two intersecting curves in space.

Inversion in a sphere sends spheres or planes to spheres or planes (p. 134). If two spheres intersect either inside or outside of a sphere of inversion K, their tangents at points of intersection are intersected along their length by planes passing through the center of K so that every tangent lies in such a plane. Since inversion does not change the angles of the planes that pass through the tangents, and since the inverted tangents must be intersected along their lengths by the original interesecting planes, the magnitudes of the tangent angles in different planes are preserved. Inversion also preserves the magnitude of angles within planes. Since inversion in a sphere preserves the magnitudes of angles within planes passing through the center of inversion and it preserves magnitudes of angles between planes passing through the center, it preserves the magnitudes of angles in three dimensions. Hence it preserves the magnitudes of angles contained by any two intersecting curves in space.

(iv) Deduce that stereographic projection is conformal.

Stereographic projection is inversion restricted to $\mathbb{C}$ or to a sphere Σ with a sphere of inversion K of radius $\sqrt{2}$, centered at N, and intersecting $\mathbb{C}$ at the equator (p. 142 and Figure [21]). Stereographic projection preserves the magnitude of angles (p. 141, bottom and Figure [20] and (iii) above). Stereographic projection preserves orientation of angles when a projection is viewed from inside the sphere (p. 142). From (ii) above, we can see that the orientation of the projection sphere Σ does not matter. Since stereographic projection preserves orientations and magnitudes of angles, it has twist.

A stereographic mapping of $\mathbb{C}$ to Σ or Σ to $\mathbb{C}$ can be regarded as analogous to a mapping of a line to a circle. Since it preserves circles and spheres within its restricted ranges, we deduce that it has uniform amplification on all ϵ vectors emanating from a single point. Since it has twist and amplification, it is conformal.