Needham, Chapter 4, Exercise 20, p. 214-215.
G. Palmer, November 29, 2015

20. The figure below is a copy of [9], p. 62.

(i) Show geometrically that if z moves distance ds along the lemniscate, increasing $\boldsymbol{\theta}$ by d$\boldsymbol{\theta}$, then w = z^2 moves a distance of 4d$\boldsymbol{\theta}$ along the circle.

We want to calculate ϕ in Figure 1(b). Then $d\tilde{s} = \phi\, r = \phi$, because r = 1. See the blue arrow for an approximation of $d\tilde{s}$. Triangles 0 w_1c and 0 w_0c are isosceles, each having two equal sides and two equal angles. Then for triangle 0 w_1c,

$$\angle\, 0\, cw_1 = \text{Pi} - 2(2\theta+2d\theta)$$

and for triangle 0 w_2 c,

$$\angle\, 0\, cw_0 = \text{Pi} - 4\theta.$$

$$\phi = |\angle\, 0\, cw_0 - \angle\, 0\, cw_1|$$

$$= |\text{Pi} - 4\theta - (\text{Pi} - 2(2\theta+2d\theta))\,|$$

$$= |\text{Pi}-\text{Pi}-4\theta+4\theta+4d\theta|$$

$$= 4d\theta$$

(ii) Using the fact that $\left(z^2\right)' = 2z$, deduce geometrically that ds = 2d$\boldsymbol{\theta}$/r.

I consulted Vasco's answer before writing this one.

The amplification of z^2 is calculated as $\left|\,(z^2)'\right|$.

$$\left|\,(z^2)'\right| = |2z| = \left|2\, r\, e^{i\theta}\right| = 2r$$

If the ϵ vector pointing along the lemniscate has length ds = r dθ, then the corresponding vector pointing along the w circle has length

$$2r\, ds = d\tilde{s} \simeq 4d\theta \quad \text{from (i)}$$

so

$$ds = 2d\theta/r$$

(iii) Using the fact that r^2 = 2 cos(2$\boldsymbol{\theta}$), show by calculation that

$$\mathbf{r\, dr = 2\sqrt{1-(r^4/4)}\, d\boldsymbol{\theta}.}$$

$$\cos(2\theta) = r^2/2$$

$\sin^2(2\,\theta) + \left(r^2/2\right)^2 = 1$

$\sin^2(2\,\theta) = 1 - r^4/4$

$\sin(2\theta) = \pm\sqrt{1-\left(r^4/4\right)}$

Take the derivative of both sides of $r^2 = 2\cos(2\theta)$.

$\left(r^2\right)' = (2\cos(2\theta))'$

2r dr = | 2*(-2) sin(2θ) dθ| This appears to be equivalent to making a choice of sign for $\sqrt{1-\left(r^4/4\right)}$.

r dr = 2 sin(2θ) dθ

$= 2\sqrt{1-\left(r^4/4\right)}\,d\theta$

(iv) Let s represent the length of the segment of the lemniscate connecting the origin to the point z. Deduce from the previous two parts that

$$\mathbf{s = \int_0^r \frac{dr}{\sqrt{1-(r^4/4)}}}$$

$r\,dr = 2\sqrt{1-\left(r^4/4\right)}\,d\theta$ from (iii)

$d\theta = \frac{r\,dr}{2\sqrt{1-(r^4/4)}}$

ds = 2dθ/r from (ii)

$= 2\left(\frac{r\,dr}{2\sqrt{1-(r^4/4)}}\right)/r = \frac{dr}{\sqrt{1-(r^4/4)}}$ by substitution for dθ

Therefore

$$s = \int_0^s ds = \int_0^r \frac{dr}{\sqrt{1-(r^4/4)}}$$

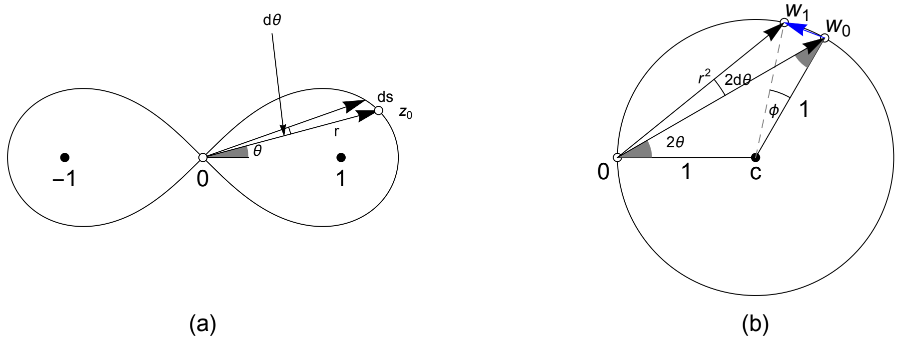

Figure 1. Derivatives of a quadratic on a lemniscate