Needham, Chapter 4, Exercise 19, p. 214.
G. Palmer, November 23, 2015
Revised after comments from Vasco.

Consider the mapping f(z) = z^4 illustrated above. On the left is a particle p travelling upwards along a segment of the line x = 1, while the right is the image path traced by f(p).

(i) Copy this diagram, and by considering the length and angle of p as it continues on its upward journey, sketch the continuation of the image path. See Figure 1.

As p continues upward, the angle θ between the ray to p and the y axis increases and approaches Pi/2 as Im[p] goes to infinity. If we write $z^4 = r^4 e^{i4\theta}$, we see that the image of p makes one revolution around the origin between θ = 0 and Pi/2, because 4(Pi/2) = 2Pi.

Then if we write $z^4 = (1 + iy)^4$, we see that as Im[p] goes to ∞, Re[$\tilde{p}$] goes to ∞^4 and Im[$\tilde{p}$] goes to $4\,\infty^3$.

$$\mathrm{Lim}_{\mathrm{Im}[p]\to\infty}(z^4) = (1 + i\infty)^4 = (1 + 2i\infty - \infty^2))^2 \;\; \text{->} \;\; \infty^4 - 4\,\infty^3\, i$$

By writing

$$z^4 = u + iv = (x^4 - 6\,x^2\,y^2 + y^4) + i\,\left(4\,x^3\,y - 4\,x\,y^3\right)$$

we also see that the path of p crosses the y axis when $(x^4 - 6\,x^2\,y^2 + y^4) = 0$, so $1 - 6\,y^2 + y^4 = 0$, and

$$y^2 = \frac{6\pm\sqrt{36-4}}{2} = \frac{6\pm\sqrt{32}}{2} = \frac{6\pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}$$

Then $y = \pm\sqrt{3 \pm 2\sqrt{2}}$.

This would lead to four values of z^4. Since we are interested only in the upward path of p and the image path $\tilde{p}$ beginning at 1 and looping up and then downward (curving positively, counterclockwise) in Figure 1, we choose $y = \sqrt{3 - 2\sqrt{2}}$, which is Im[p] when θ = Pi/8 and $\tilde{p}$ = 1.37258 i (at A), and we choose $y = \sqrt{3 + 2\sqrt{2}}$, which is Im[p] when θ = 3Pi/8 and $\tilde{p}$ = -46.6274i (See red dots in Figure 1).

Figure 1 shows the path of the image of p curving in a positive direction down with acceleration towards the origin, but the path goes to powers of infinity on both the x and y axes. If we were to take the path of p from Im[p] = -∞ to ∞, or θ = -Pi/2 to Pi/2, then we could see that it is possible for the image of p to make two full loops around the origin. As it is, we see only one partial loop. In Figure 2 b, we see two half loops.

(ii) Show that A = i $\sec^4$(Pi/8).

The following are true at any point z = x + iy. See Figures 2a and 2b.

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = 1^2 - r^2 \sin^2 \theta$$

$$r^2 + r^2 \sin^2 \theta = 1$$

$$r^2 (1 + \sin^2 \theta) = 1$$

$$r^2 = 1 \Big/ \left(1 + \sin^2 \theta\right)$$

$$\mathrm{r} = \pm \frac{1}{\sqrt{1-\sin^2 \theta}} = \sec \theta \quad \text{[trig identity]} \qquad (1)$$

Let $z = re^{i\theta}$. Then

by taking the 4th power

$$A = z^4 = r^4 e^{i4\theta} \qquad (2)$$

by substitution

$$A = \sec^4(\theta)\, e^{i(\arg(A))} \qquad (3)$$

by construction, arg(A) = Pi/2

$$A = \sec^4(\theta)\, e^{i(\mathrm{Pi}/2)} \qquad (4)$$

Set $\arg(z^4)$ in (2) equal to $\arg(\sec^4(\theta)\, e^{i(\mathrm{Pi}/2)})$ in (4).

$$4\theta = \mathrm{Pi}/2 \quad \Longrightarrow \quad \theta = \mathrm{Pi}/8$$

Since arg(A) = Pi/2 and $e^{i(\mathrm{Pi}/2)} = \mathrm{i}$, $A = i \sec^4(\mathrm{Pi}/8)$.

Out[2545]=

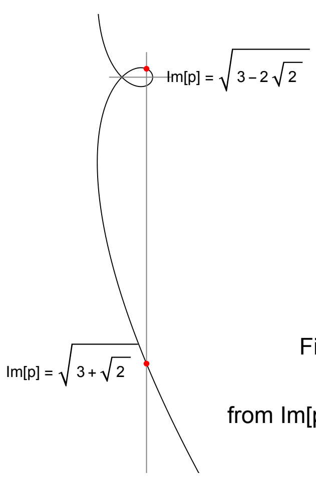

Figure 1. Continuation of z^4 on x = 1

from Im[p] < $\sqrt{3-2\sqrt{2}}$ to Im[p] > $\sqrt{3+2\sqrt{2}}$

Out[2573]=

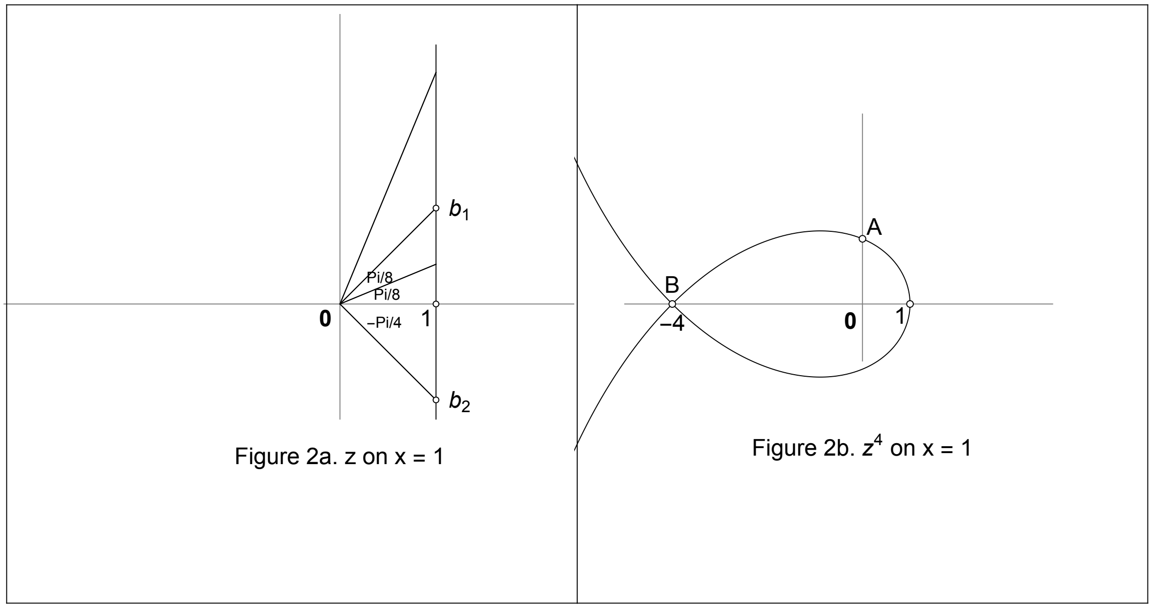

Figure 2a. z on x = 1

Figure 2b. z^4 on x = 1

(iii) Find and mark on your picture the two positions (call them b_2 and b_2) of p that map to the self-intersection point B of the image path.

z^4 = u because vi = 0

Let z = x + iy.

$u = \mathrm{Re}[(x + iy)^4] = x^4 - 6 x^2 y^2 + y^4$

$v = \mathrm{Im}[(x + iy)^4] = 4 x^3 y - 4 x y^3 = 0 \Longrightarrow y = \pm 1$, because x = 1

The two points must be $1 \mp i$ (Figure 2a).

$b_1 = 1 + \mathrm{i}$

$b_2 = 1 - \mathrm{i}$

$\mathrm{B} = (1 \pm i)^4 = -4$

This value can also be obtained by substituting ±1 into the formula for u.

(iv) Assuming the result f'(z) = 4 z^3, find the twist at b_1 and also at b_2.

We can see from Figure 2a, that θ at b_1 is Pi/4 (or we could calculate it from arctan(1) = Pi/4).

The twist is arg(f'(z)) $= \arg\left(4 \left| z \right| e^{\mathrm{i}3\theta}\right) = 3\theta = \pm 3\mathrm{Pi}/4$.

(v) Using the previous part, show that (as indicated at B) the image path cuts itself at right angles.

Since z^4 is analytic and conformal, the twist and the argument of the tangent are the same: arg($\left(z^4\right)'$ at z = (1±i)). The tangents in this case have the same angles as the twist, ±3Pi/4. Angle -3Pi/4 is equivalent to 5Pi/4. Then

5Pi/4 - 3Pi/4 = Pi/2, which is a right angle.

Other observations not bearing directy on the questions.

Figure 3 plots ϵ vectors emanating from B at tangent angles. Multiplying both ϵ tangents by the amplitwist illustrates that z^4 is conformal.

Out[2598]=

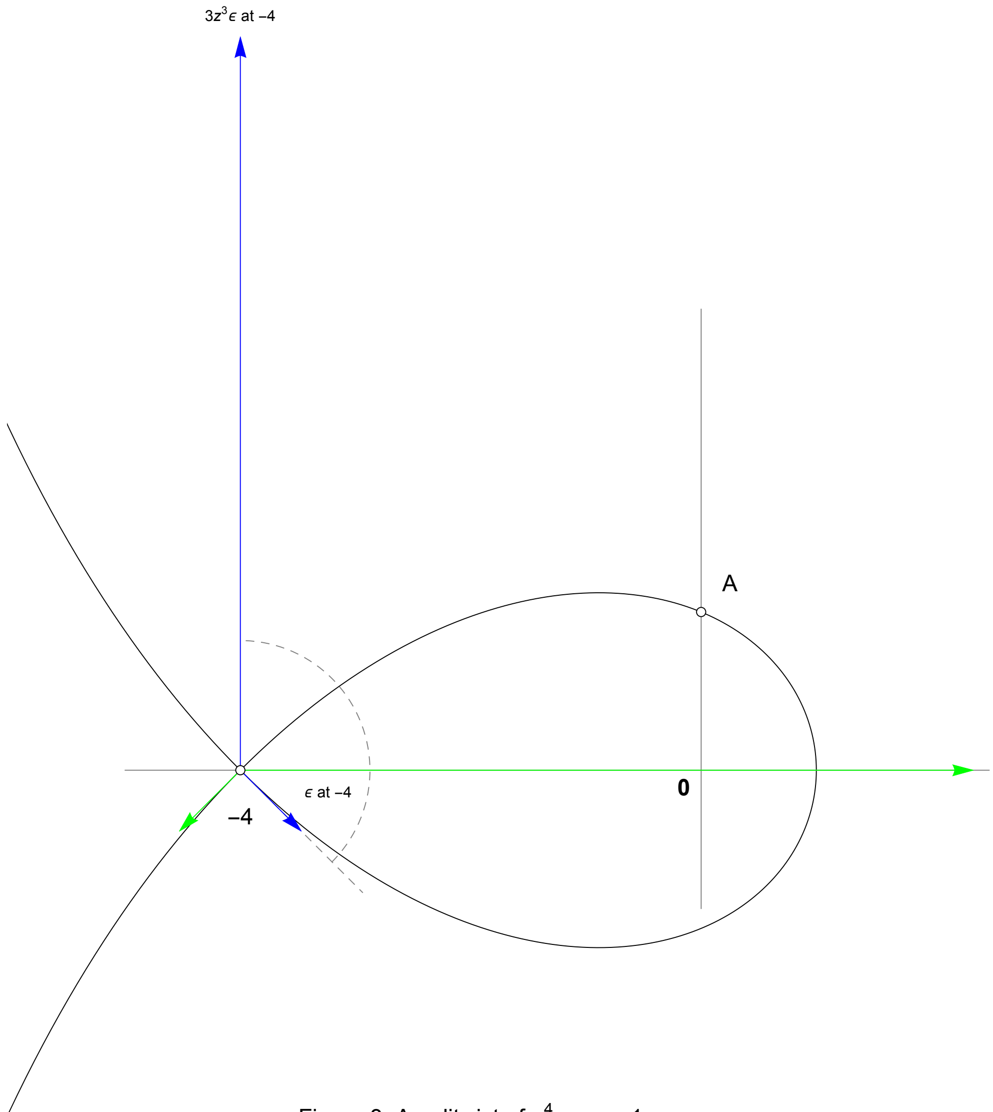

Figure 3. Amplitwist of z^4 on x = 1