Needham, Chapter 4, Exercise 18, p. 213.
G. Palmer, November 21, 2015

18. Recall Ex. 19, p. 186, where you showed that a general Möbius transformation

$$M(z) = \frac{az+b}{cz+d}$$

maps concentric circles to concentric circles if and only if the original circle (call if $\mathcal{F}$) is centred at q = -d/c. Let $\rho = |z-q|$ be the distance from q to z, so that members of $\mathcal{F}$ are $\rho \equiv$ const.

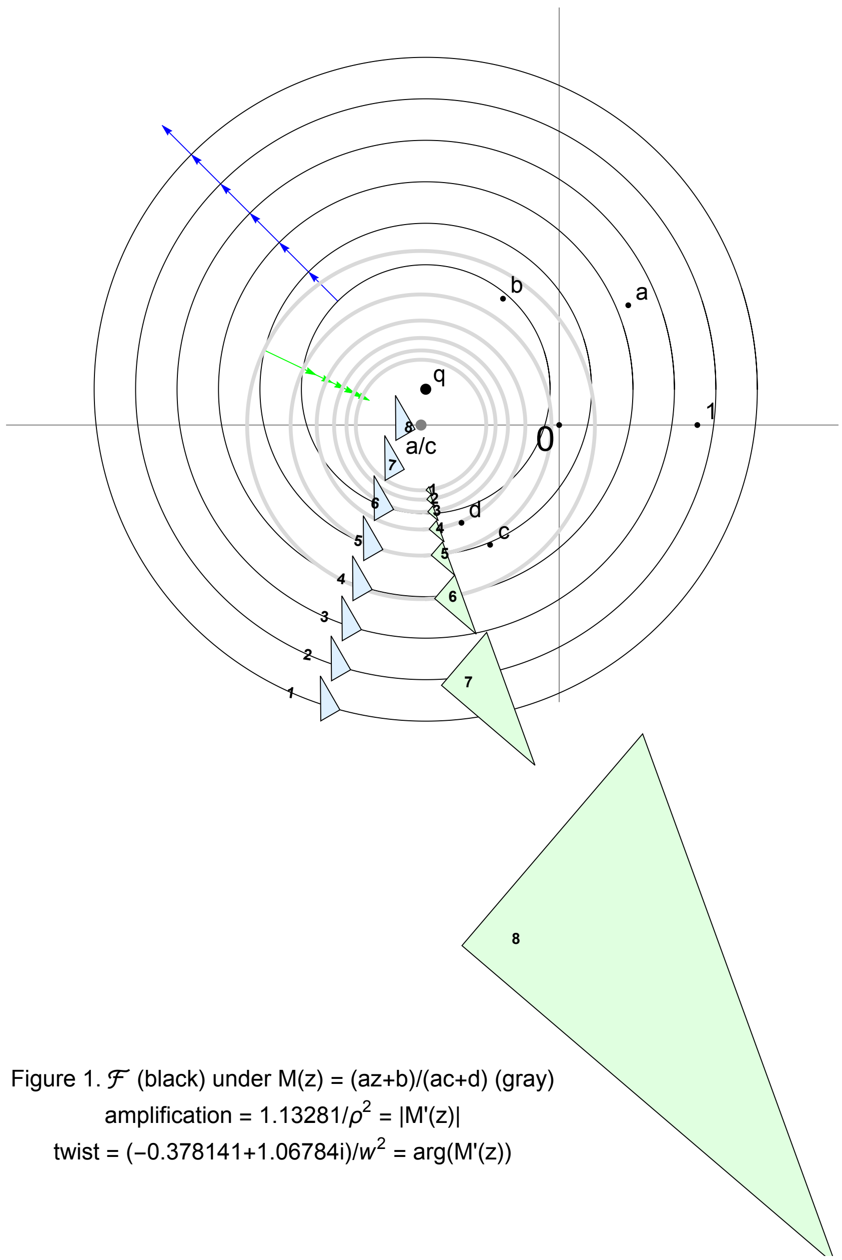

Figure 1. $\mathcal{F}$ (black) under M(z) = (az+b)/(ac+d) (gray)
amplification = $1.13281/\rho^2$ = |M'(z)|
twist = $(-0.378141+1.06784i)/w^2$ = arg(M'(z))

(i) By considering orthogonal connecting vectors from one member of $\mathcal{F}$ to an infinitesimally larger member of $\mathcal{F}$, deduce that the amplification of M is constant on each circle of $\mathcal{F}$. Deduce

that |M'| must be a function of ρ alone.

The circles of $\mathcal{F}$ are concentric, so orthogonal ϵ arrows connecting $\mathcal{F}_n$ to $\mathcal{F}_{n-1}$ have the same length on the entire circumference of $\mathcal{F}_n$. M preserves angles, so an orthogonal ϵ arrow on $\mathcal{F}$ goes to an orthogonal ϵ arrow on $\tilde{\mathcal{F}}$.

Let w = M(z) = $\tilde{\rho}\, e^{i\theta}$. We can expect $\tilde{\rho}$ to be constant across all θ because the circles of $\tilde{\mathcal{F}}$ are concentric. We also expect $\tilde{\epsilon}$ to be constant across all θ for the same reason. If $\tilde{\rho}_\theta$ were different from $\tilde{\rho}_{\theta+\phi}$ on a circle of $\tilde{\mathcal{F}}$, then orthogonal angles would not be preserved. Hence, the amplification of M is constant on each circle of $\mathcal{F}$ and |M'| must be a function of ρ alone, as opposed to ρ and θ.

We attempt to deduce this by taking the derivative of M(z).

$$M'(z) = \left(\frac{az+b}{cz+d}\right)' = \left((az+b)(cz+d)^{-1}\right)'$$

$$= a(cz+d)^{-1} + (az+b)(-c)(cz+d)^{-2}$$

$$= \frac{a(cz+d)-c(az+b)}{(cz+d)^2} = \frac{acz+ad-(acz+bc)}{c^2(z+q)^2} = \frac{ad-bc}{c^2(z+q)^2} \qquad (1)$$

We use (1) in (vi) and in programming twist in Figure 1.

$$|M'(z)| = \frac{|ad-bc|}{\left|c^2(z+q)^2\right|} = \frac{|ad-bc|}{|c|^2\rho^2}$$

Points a, b, c and d are constants. Hence, for M such that (ad-bc) ≠ 0 and c ≠ 0, |M'| depends on ρ alone.

(ii) By considering the image of an infinitesimal shape that starts far from q and then travels to a point very close to q, deduce that at some point in the journey, the image and preimage are congruent.

[November 17, 2015. I think there is a problem with the question. Given the distinction between image and preimage, if the image is M($\mathcal{F}$), then it does not naturally travel to a point very close to q by decreasing ρ, which is a quality of the preimage, but the image may approach a/c with increasing ρ. Therefore, I think the question should read *"By considering the image under M of an infinitesimal shape that starts far from q and then travels to a point very close to q, deduce that at some point in the journey, the image and preimage are congruent."]*

Stipulate that the image is M(P), where P is the preimage, z is a point in the preimage, and ρ = |z-q| = |z+d/c|. Let ϵ be the vector of any side of the preimage or a tangent emanating from a point on the preimage and $\tilde{\epsilon}$ be the corresponding side or tangent on the image. M is conformal, because it is a Möbius transformation. Angles are preserved. It has amplitwist because it is conformal over a region. Since it has amplitwist, the image at any size is similar to the preimage. We have shown in (i) that the amplification |M'| depends only on ρ. Furthermore, Vasco has called attention to the fact that as ρ decreases, $\tilde{\epsilon}$ vectors increase (See (i)). In this case, we are interested in decreasing ρ. When ρ is very large, $|\tilde{\epsilon}|$ is very near zero. When ρ is very small, $|\tilde{\epsilon}|$ is large. During the journey, which is measured by decreasing ρ, $|\tilde{\epsilon}|$ increases until $|\tilde{\epsilon}| = |\epsilon|$, at which point $|M'(z)| = 1$ for all z, and the images are congruent.

Discussion of Figure 1: Figure 1 illustrates these points. The triangles represent ϵ shapes. The outermost preimage triangle (blue) corresponds to the innermost image triangle (green). Because M is conformal, these images are similar. As ρ decreases for the preimage triangles, $\tilde{\rho} = |\tilde{z}\text{-}(a/c)|$ increases for the image triangles and they increase in size. At about the second $\tilde{\mathcal{F}}$ image circle from the center, the image triangle appears to be nearly congruent with the preimage triangles. This is where $|M'| \approx 1$. We have calculated the twist as $\arg(M'(z)) = \arg(\,(ad - bc)/(c^2 w^2)\,)$ without normalization. Both the preimage and image triangles were rotated by Pi/2 around q and a/c respectively after the calculations just to get them out of the way of the ϵ arrows, but by the question, they can be situated at any arg(z) and arg(M(z)).

Vasco also pointed out that the twist can be rewritten as $\arg(A/w^2)$, where $A = (ad - bc)/c^2) = ge^{i\alpha}$, so $\arg(A/w^2) = -2\phi + \alpha$ and $\phi = \arg(w)$. We can also see by Figure[1], p. 124, and by obtaining step (iii) in (3) on the same page, that M(z) has the rotation $\arg(-A/w) = -(\alpha - 2\phi) = -2\phi + \alpha + \text{Pi}$. (Step 4 would not alter the rotation or twist.) This shows that the twist of M reverses the direction of the arrows.

(iii) Combine the above results to deduce that there is a special member I_M of $\mathcal{F}$ such that infinitesimal shapes on I_M are mapped to congruent image shapes on the image circle $M(I_M)$. Recall that I_M is called the isometric circle of M.

I_M is the member at which $|M'(z)| = 1$ for all z in I_M.

(iv) Use the previous part to explain why $M(I_M)$ has the same radius as I_m.

The amplification depends only on ρ. Since $|\tilde{\epsilon}| = |\epsilon|$, it must be the case that $M'(z) = M'(w) = 1$, which implies that $(w = M(z)) = z$, and since $\rho = |z\text{-}q|$ and $\tilde{\rho} = |w\text{-}q|$, it follows that $\tilde{\rho} = \rho$.

(v) Explain why $I_{M^{-1}} = M(I_M)$.

$I_M = M(I_M)$, from (iv), because they have the same radius.

$I_{M^{-1}} = M^{-1}(I_M) = M^{-1} \circ M(I_M) = I_M = M(I_M)$

(vi) Suppose that M is normalized. Using the idea in Ex. 16, show that the amplification of M is

$$|M'(z)| = \frac{1}{|c|^2 \rho^2}$$

From Ex. 16, the amplification a = |f'(z)|.

$$|M'(z)| = \frac{|ad - bc|}{|c|^2 \rho^2} \qquad \text{from (i)}$$

If M is normalized, ad-bc = 1. Then, letting $k = \pm\sqrt{ad - bc}$, there is a $\left|\frac{c}{k}\right|^2$ in the denominator, so the k goes to 1.

$$|M'(z)| = \frac{1}{|c|^2\rho^2}$$