Needham, Chapter 4, Exercise 17, p. 213.

G. Palmer, November 10, 2012

17. Consider the complex inversion mapping I(z) = (1/z).

(i) If z = x + iy and I = u + iv, express u and v in terms of x and y.

$$1/z = 1/(x+iy) = (x - iy)\big/\left(x^2 + y^2\right) = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

Then u = $\frac{x}{x^2+y^2}$ and v = $\frac{-y}{x^2+y^2}$.

(ii) Show that the Cauchy-Riemann equations are satisfied everywhere except the origin, so that I is analytic except at this point.

$$\partial_x u = \partial_x x\,\frac{1}{x^2+y^2} + x\,\partial_x \frac{1}{x^2+y^2}$$

$$= \frac{1}{x^2+y^2} + x(-1)\left(x^2+y^2\right)^{-2}2x$$

$$= \frac{x^2+y^2}{\left(x^2+y^2\right)^2} - \frac{2\,x^2}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\text{, undefined at origin}$$

$$\partial_y v = \frac{-1}{x^2+y^2}\ \text{-y(-1)}\left(x^2+y^2\right)^{-2}2y$$

$$= \frac{-\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2} + \frac{2\,y^2}{\left(x^2+y^2\right)^2} = \frac{y^2-x^2}{\left(x^2+y^2\right)^2}\text{, undefined at origin}$$

The first CR equation is satisfied everywhere but at the origin: $\partial_x u = \partial_y v$.

$$\partial_x v = \partial_x(-y)\,\frac{1}{x^2+y^2}\ \text{-y}\ \partial_x \frac{1}{x^2+y^2}$$

$$= 0 - y((-1)\left(x^2+y^2\right)^{-2}2x$$

$$= \frac{2\,xy}{\left(x^2+y^2\right)^2}\text{, undefined at origin}$$

$$\partial_y u = \partial_y x\,\frac{1}{x^2+y^2} + x\,\partial_y \frac{1}{x^2+y^2}$$

$$= 0 + x(-1)\left(x^2+y^2\right)^{-2}2y$$

$$= \frac{-2\,xy}{\left(x^2+y^2\right)^2}\text{, undefined at origin}$$

The second CR equation is satisfied everywhere but at the origin: $\partial_x v = -\partial_y u$.

(iii) Find the Jacobian matrix, and by expressing it in terms of polar coordinates, find the local geometric effect of I.

$$J = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2\,xy}{(x^2+y^2)^2} \\ \frac{2\,xy}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{r^2\sin^2\theta - r^2\cos^2\theta}{r^4} & \frac{-2\,r^2\cos\theta\sin\theta}{r^4} \\ \frac{2\,r^2\cos\theta\sin\theta}{r^4} & \frac{r^2\sin^2\theta - r^2\cos^2\theta}{r^4} \end{pmatrix}$$

$$= \frac{1}{r^2}\begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$= -\frac{1}{r^2}\begin{pmatrix} \cos(-2\,\theta) & -\sin(-2\,\theta) \\ \sin(-2\,\theta) & \cos(-2\,\theta) \end{pmatrix} \qquad (1)$$

The local geometric effect on ϵ (infinitesimal) arrows is rotation by -(2θ + Pi) (clockwise) and amplification by $\frac{1}{r^2}$. Figure 1 illustrates the effect of J on the blue and green arrows.

(iv) Use (7) to show that the amplitwist is $-(1/z^2)$, just as in ordinary calculus, and in accord with the previous exercise. Use this to confirm the result of part (iii).

We let $z = re^{i\theta}$.

$$f' = \partial_x u + i\,\partial_x v = \partial_x f \qquad (7)$$

$$= -\frac{1}{r^2}(\cos(-2\theta) + i\sin(-2\theta)) \qquad \text{(from (iii, (1)))}$$

$$= -\frac{1}{r^2}e^{-i2\theta} = -\frac{1}{r^2\,e^{2\,\theta}} = -1/z^2 = -(1/z^2) \text{ (Euler's equation)}$$

This shows again that the local geometric effect of **I**(z) = 1/z on ϵ arrows is rotation by -(2θ+Pi) and amplification by $\frac{1}{r^2}$, where r = |z|.

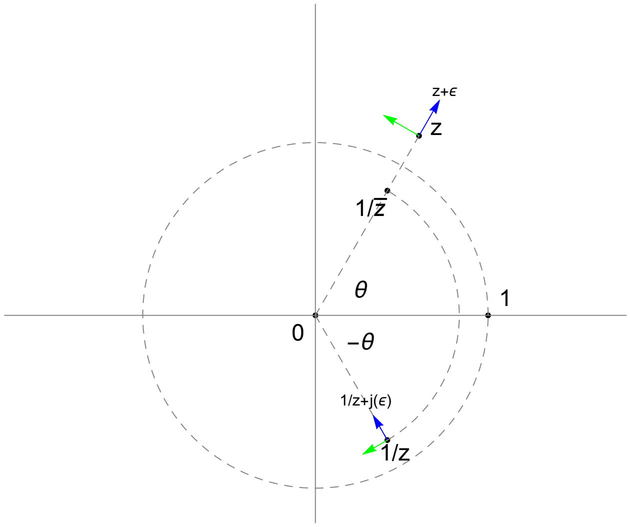

Figure 1. Amplitwist with Rotation by Jacobian J