Needham, Chapter 4, Exercise 16, p. 213.
G. Palmer, November 11, 2012

16. Consider the complex inversion mapping I(z) = (1/z). Since I is conformal, its local effect must be an amplitwist. By considering the image of an arc of an origin-centred circle, deduce that |(1/z)'| = 1/ | z |^2.

This is rewritten after comments from Vasco.

By inspection of Figure 1, the *amplification* of 1/z equals the contraction of $1/\overline{z}$ (cf. Figure [1a], p. 124]). This amplification can be written $|1/\overline{z}|/|z| = |1/z|/|z| = \frac{1}{|z|^2}$. Figure 1 also shows that if arg(z) = θ, then arg(1/z) = -θ, so the twist is -2θ. In this case, it seems valid to write the amplitwist as

$$ae^{i\alpha} = \frac{1}{|z|^2}e^{-i2\theta},$$

which is a twist followed by an amplification. Compare z ↦ Az, p. 199. **This is wrong**. The twist is not always seen in the rotation of f(z) on z. It is seen in Figure 1 in the rotation of the ϵ (infinitesimal) arrows, which have a different rotation from f(z). The twist of the ϵ arrows is actually -(2θ+Pi), so we write

$$ae^{i\alpha} = \frac{1}{|z|^2}e^{-i(2\theta+\mathrm{Pi})} = \frac{1}{|z|^2}\,e^{-i\mathrm{Pi}}e^{-i2\theta} = -\,\frac{1}{|z|^2}e^{-i2\theta}$$

This is the same expression that we get when we take the derivative of 1/z, where z = $|z|\,e^{i\theta}$. Since the function is conformal and analytic,

$$(1/z)' = \left(z^{-1}\right)' = -z^{-2} = -\,\frac{1}{z^2} = -\,\frac{1}{|z|^2}e^{-i2\theta}$$

With this approach, we can understand why Needham specified the absolute value of (1/z)'.

$$|(1/z)'| = \left|\frac{-1}{|z|^2}e^{-i2\theta}\right| = \frac{1}{|z|^2}$$

Compare p. 198 where $f'(z) = |f'(z)|\,e^{i\arg[f'(z)]}$.

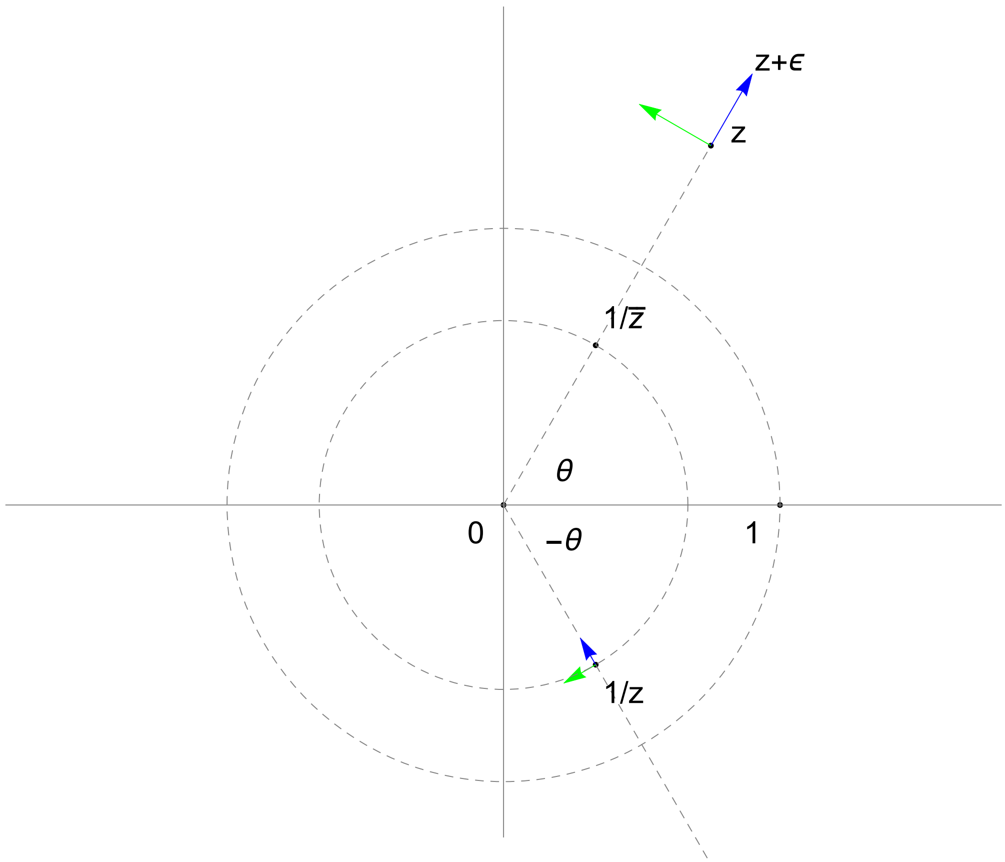

Figure 1. $|(1/z)'| = 1/|z|^2$