Needham, Chapter 4, Exercise 15, pp. 212-213.
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15. Let us define S to be the square region given by

$$a - b \le \mathrm{Re}(z) \le a + b \quad \text{and} \quad -b \le \mathrm{Im}(z) \le b.$$

(i) Sketch a typical S for which b < a. Now sketch the image $\tilde{S}$ under which $z \mapsto e^z$.

See Figure 1.

(ii) Deduce the area of $\tilde{S}$ from your sketch, and write down the ratio

$$\Lambda = \left(\frac{\text{area of } \tilde{S}}{\text{area of } S}\right).$$

Let A, B, C, and C be the four vertices of S taken clockwise. Then, letting A_S be the area of S,

A = (a-b) + ib
B = (a+b) + ib
C = (a+b) - ib
D = (a-b) - ib

$$A_s = |A - D|^2 = (2b)^2 = 4b^2$$

e^z takes vertical lines to circles and horizontal lines to rays emanating from the origin. The vertical sides of S go to arcs of concentric circles with radii e^b for the inner arc and e^{a+b} for the outer arc. If we extend the arcs to circles labeled “inner” and “outer”,

$$A_{\text{inner}} = \text{Pi}\left(e^{a-b}\right)^2 = \text{Pi}\ e^{2(a-b)}$$

$$A_{\text{outer}} = \text{Pi}\left(e^{a+b}\right)^2 = \text{Pi}\ e^{2(a+b)}$$

The area of the annulus of their difference is

$$A_{\text{annulus}} = A_{\text{outer}} - A_{\text{inner}}$$

$$= \text{Pi}\left(e^{2(a+b)} - e^{2(a-b)}\right)$$

$$= \text{Pi}\left(e^{2a} e^{2b} - e^{2a}/e^{2b}\right)$$

$$= \text{Pi}\ e^{2a}\left(e^{2b} - e^{-2b}\right)$$

The angle $\theta_{\tilde{S}}$ between the two rays of the annulus extended to their intersection at zero is calculated without proving that they intersect at zero, though it would not be hard to show that a horizonal line goes to a ray. $e^z = e^{x+iy} = e^x e^{iy}$ does not change it’s angle as x increases.

$$\theta_{\tilde{S}} = \arg(f(B)\text{-}f(A)) - \arg(f(C)\text{-}f(D))$$

$$= \arg\left(\frac{f(B)-f(A)}{f(C)-f(D)}\right) = \arg\left(\frac{e^{(a+b)+ib}-e^{(a-b)+ib}}{e^{(a+b)-ib}-e^{(a-b)-ib}}\right)$$

$$= \arg\left(\frac{e^{(a+b)}\,e^{ib}-e^{(a-b)}\,e^{ib}}{e^{(a+b)}\,e^{-ib}-e^{(a-b)}\,e^{-ib}}\right) = \arg\left(\frac{e^{ib}\left(e^{(a+b)}-e^{(a-b)}\right)}{e^{-ib}\left(e^{(a+b)}-e^{(a-b)}\right)}\right)$$

$$= \arg\left(e^{2\,ib}\right) = 2b$$

Calculate the area of $\tilde{S}$ as the fraction b/Pi of the area of the annulus.

$$A_{\tilde{S}} = \frac{\theta_{\tilde{S}}}{2\,\text{Pi}}\,A_{\text{annulus}} = \frac{2\,b}{2\,\text{Pi}}\text{Pi}\;e^{2\,a}\left(e^{2\,b} - e^{-2\,b}\right)$$

$$= \text{b}\;e^{2\,a}\left(e^{2\,b} - e^{-2\,b}\right)$$

$$\Lambda = \left(\frac{\text{area of }\tilde{S}}{\text{area of }S}\right)$$

$$= \frac{b\;e^{2\,a}\left(e^{2\,b}-e^{-2\,b}\right)}{4\,b^2} = \frac{e^{2\,a}\left(e^{2\,b}-e^{-2\,b}\right)}{4\,b}$$

(iii) Using the results of the previous two exercises, what limit should Λ approach as b shrinks to nothing?

From Ex. 13, the amplification of $(e^z)'$ is e^x. From Ex. 14, the local magnification factor is the square of the amplification, $e^{2\,x}$.

$$\lim_{b\to 0}(\Lambda) = (\lim_{b\to 0}\,(e^x))^2$$

$$= (e^a)^2 = e^{2\,a}$$

(iv) Find $\lim_{b\to 0}(\Lambda)$ from your expression in part (ii) and check that it agrees with your geometric answer in part (iii).

$$\frac{e^{2\,a}\left(e^{2\,b}-e^{-2\,b}\right)}{4\,b} = \frac{e^{2\,a}}{2}\,\frac{\left(e^{2\,b}-e^{-2\,b}\right)}{2\,b}$$

$$\lim_{b\to 0}(\Lambda) = \lim_{b\to 0}\left[\frac{e^{2\,a}}{2}\,\frac{\left(e^{2\,b}-e^{-2\,b}\right)}{2\,b}\right] = \lim_{b\to 0}\left[\;\frac{e^{2\,a}}{2}\;\left(\frac{e^{2\,b}}{2\,b}\text{-}\frac{e^{-2\,b}}{2\,b}\right)\right]$$

Use the power series for e^z from p. 79 to find $\frac{e^{2\,b}}{2\,b}$ and $\frac{e^{-2\,b}}{2\,b}$.

$$\frac{e^{2\,b}}{2\,b} = \left(1 + 2\,b + 4\,b^2/2 + 8\,b^3\big/6 + \ldots\right)/2\text{b}$$

$= 1/(2b) + 1 + b + 2\,b^2/3 + \ldots$

$\frac{e^{-2b}}{2b} = (1\text{-}2b + 4\,b^2/2 - 8\,b^3/6 + \ldots)/2\,b$

$= 1/(2b) - 1 + b - 2\,b^2/3 + \ldots$

$\frac{e^{2b}}{2b} - \frac{e^{-2b}}{2b} = 0 + 2 + 0 + 4\,b^2/3 + 0 \ldots$

$\lim_{b\to 0}[\frac{e^{2b}}{2b} - \frac{e^{-2b}}{2b}] = \lim_{b\to 0}[\,2 + 4\,b^2/3 + \ldots] = 2$

(because all the terms after the first contain a power of b)

$\lim_{b\to 0}(\Lambda) = \lim_{b\to 0}[\,\frac{e^{2a}}{2}\,(\frac{e^{2b}}{2b} - \frac{e^{-2b}}{2b})] = \lim_{b\to 0}[\,\frac{e^{2a}}{2}\,2]$

$= e^{2a}$

The limit of Λ as b → 0 equals the magnification of A_S. Very nifty.

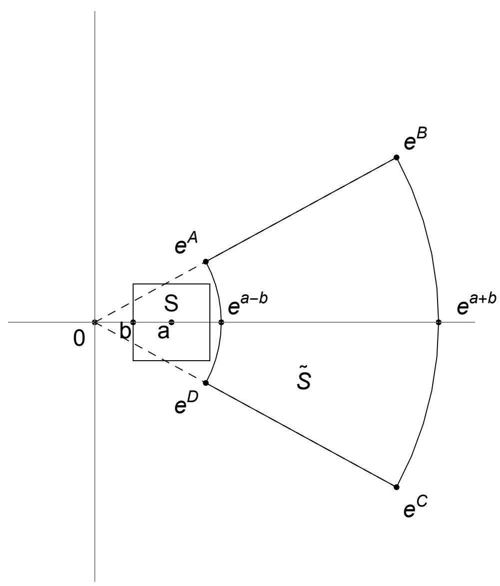

Figure 1. Square under e^z