Needham, Chapter 4, Exercise 14, p. 212.
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14. By sketching the image of an infinitesimal rectangle under an analytical mapping, deduce that the local magnification factor for area is the square of the amplification. Rederive this fact by looking at the determinant of the Jacobian matrix.

Amplification increases or shrinks vectors such as the sides of the infinitesimal "ϵ rectangle" by equal factors. Since area is the product of two sides, the change in area is the product of the change in length of two sides. Since the derivatives of an analytical mapping apply equally to all sides of the ϵ rectangles, the magnification factor for the area is the square of the amplification.

Rederive this fact by looking at the determinant of the Jacobian matrix.

We begin with the bottom of p. 199, where the amplitwist is expressed as (Az)' = A, where A = a $e^{i\alpha}$, which "represents the combination of an origin centered expansion by a, and a rotation by α." Multiplying by A is equivalent to multiplying by the derivative.

The Jacobian is:

$$J = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

Since the mapping is analytical, it has the structure

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$ ("a" here is not the same as the "a" for expansion)

for which the determinant is $a^2 + b^2$, which gives

$$\det J = (\partial_x u)^2 + \left(\partial_y u\right)^2$$

Using u and v for the sides of the ϵ rectangles and equation (7), p. 210 for the amplitwist, we get

$$f' = \partial_x u + i\, \partial_x v = \partial_x f \qquad (7)$$

$$u_{n+1} = f'\, u_n, \qquad v_{n+1} = f'\, v_n$$

The area of the ϵ rectangle is

$$|u_{n+1}|\,|v_{n+1}| = |f'|\,|u_n|\ \ |f'|\,|v_n|$$

$$= ((\partial_x u_n)^2 + (\partial_x v_n)^2)\ \ |u_n|\,|v_n|$$

$$= (\det J)\ |u_n|\,|v_n|$$

But (det J) = $|f'|^2$ and f' = a$e^{i\alpha}$, so |f'| = a and the magnification of area is a^2, which is the square of the amplification.

Figure 1 illustrates a specific case. An ϵ rectangle which has vertices multiplied successively by the complex number (.4+.4i). That is, f(z) = (.4+.4i) z. It appears to have both twist and amplitude. Each complex side of the rectangle is multiplied by .4+.4i. Hence, if the old side was x+iy, the new side is

$$(.4+.4i)(x+iy) = .4x + .4iy + .4ix - .4y$$

$$= .4(x-y) + .4i(x+y)$$

We calculate the amplification factor a of $.4\sqrt{2}$ (See p. 199).

$$[(.4\,(x-y))^2 + (.4\,(x+y))^2]^{1/2}$$

$$= \left[.16\left(x^2 - 2\,xy + y^2 + x^2 + 2\,xy + y^2\right)\right]^{1/2}$$

$$= \sqrt{.16\left(2\,x^2 + 2\,y^2\right)} = .4\sqrt{2}\sqrt{x^2+y^2} = .4\sqrt{2}\,|z|$$

The magnification (shrinkage) factor for area should therefore be the square of the amplification, which is $\left(.4\sqrt{2}\right)^2$ = .32 times the area.

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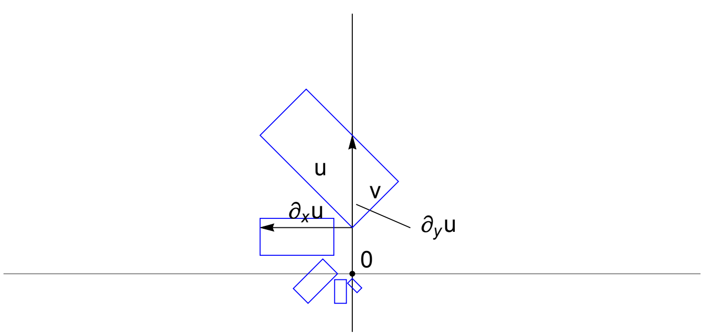

Figure 1. ϵ rectangles under analytic f(z)