Needham, Chapter 4, Exercise 12, p. 212.
G. Palmer, November 6, 2015

12. For nonzero z, let f(z) = f(x+iy) = xy/$\bar{z}$.

We have not been able to get a coherent solution to all three questions. Vasco agreed and suggested that there is an error in f(z). It should read

$$f(x+iy) = |xy|/\bar{z}$$

(i) Show that f(z) approaches 0 as z approaches any point on the real or imaginary axis, including the origin.

$$|xy|/\bar{z} = |xy|/(x-iy)$$

$$= \frac{|x|}{|z|}\frac{|y|}{|z|}(x+iy)$$

Write z = r $e^{i\theta}$

$$= |\sin\theta \cos\theta|\ (x + iy)$$

As z approaches the real axis, $\theta \to 0$.

$$f(z) = \lim_{y\to 0}|\sin\theta \cos\theta|\ (x + iy)$$

$$= |0 \cos\theta|\ x = 0$$

As z approaches the imaginary axis, $\theta \to$ Pi/2 or 3Pi/2.

$$f(z) = \lim_{x\to 0}|(\sin\theta)\ 0|\ \ iy$$

$$= |(\sin\theta)\ 0|\ \ iy = 0$$

At z = 0, we have f(z) = $|\sin\theta \cos\theta|$ * 0 = 0.

We conclude that f(z) goes to zero everywhere on the x and y axes, including z = 0.

(ii) Having established that f = 0 on both axes, deduce that the Cauchy-Riemann equations are satisfied at the origin.

We have established that f = 0 on both axis. At f(0) = 0, we can write

$$f(0) = u + v\,i = 0 + 0\,i$$

$$\partial_x u = \partial_x v = \partial_y u = \partial_y v = 0$$

Since all the partials are zero, the Cauchy-Riemann equations are satisfied.

(iii) Despite this, show that f is not even differentiable at 0, let alone analytic there! To do so, find the image of an infinitesimal arrow emanating from 0 and pointing in the direction $e^{i\phi}$. Deduce that while f does have a twist at 0, it fails to have an amplification there.

In the following, I write ϵ for ‘infinitesimal’ or ‘infinitesimal arrow’.

We can deduce that f has a twist at 0 if it rotates two ϵ arrows by the same amount. It can only do that if it has amplitwist at 0 as shown by a derivative. Since f(0) = 0, f’(0) = 0. Hence, $\epsilon_1 = 0\, e_0^{i*0} = 0 * 1 = 0$. Both ϵ arrows have zero twist and zero amplification at 0. By this arithmetic approach, f is differentiable at 0.

An alternate solution makes use of the trigonometric form of f:

$$f(\epsilon) = |\cos(\phi)\,\sin(\phi)|\;|\epsilon|\;e^{i\phi}$$

We draw two ϵ arrows, ϵ_1 and ϵ_2 emanating from the origin (Figure 1, blue). This trigonometric form shows that f does not rotate ϵ. Since both arrows undergo zero rotation, they have zero twist. $\alpha = 0$ where amplitwist is $A = ae^{i\alpha}$ (p. 199). The arrows are expanded by different amounts by $|\cos(\phi)\,\sin(\phi)|$ depending on ϕ for each arrow. For example, suppose that ϕ_1 = Pi/3 and ϕ_2 = 11Pi/12. Then the expansions (green arrows) are .43 and .25. There is no amplification, so by this geometric approach f is not differentiable or analytic at 0. Since our chapter deals with amplitwist, this seems the most appropriate answer.

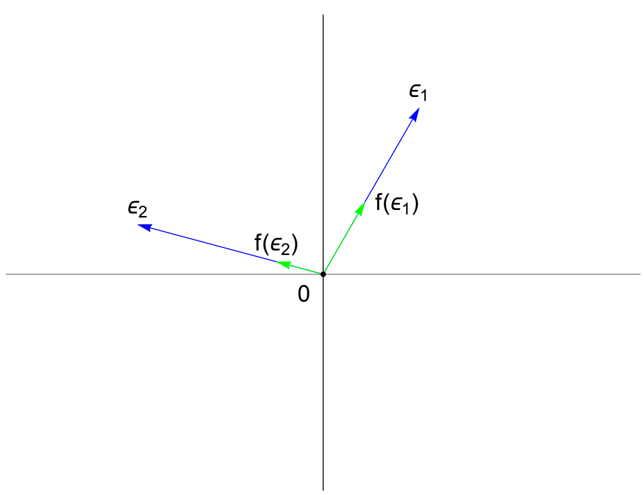

Figure 1. Infinitesimal arrows under f(z) = |cos(z)sin(z)|z