

Needham, Chapter 4, Exercise 11, p. 212.

G. Palmer, October 30, 2015

11. Let's agree to say that " $f = u + iv$ satisfies the Cauchy-Riemann equations" if u and v do. Show that if $f(z)$ and $g(z)$ both satisfy the Cauchy-Riemann equations, then their sum and their product do also.

Let $f(z) = u + iv$ and $g(z) = s + it$, where u, v, s , and t are complex functions of x and y , be analytic complex functions that satisfy the Cauchy-Riemann equations. Then

$$\partial_x u = \partial_y v, \quad \partial_x v = -\partial_y u \quad (1,2)$$

and

$$\partial_x s = \partial_y t, \quad \partial_x t = -\partial_y s \quad (3,4)$$

Test the sum:

$$f(z) + g(z) = (u+s) + i(v+t)$$

Write the CR equations. These are not assumed to be true. They are to be tested with substitutions from (1,2,3,4). If the equalities hold after the substitutions, then we conclude that the sum satisfies the CR equations.

$$\partial_x (u + s) = \partial_y (v + t) \quad (5)$$

$$\partial_x (v + t) = -\partial_y (u + s) \quad (6)$$

Expand (5) and (6)

$$\partial_x u + \partial_x s = \partial_y v + \partial_y t \quad (7)$$

$$\partial_x v + \partial_x t = -\partial_y u - \partial_y s \quad (8)$$

Substitute on the LHS from (1, 2, 3, 4):

$$\partial_y v + \partial_y t = \partial_y v + \partial_y t \quad \text{Equation 1 is satisfied.}$$

$$-\partial_y u - \partial_y s = -\partial_y u - \partial_y s \quad \text{Equation 2 is satisfied.}$$

The sum of $f(z)$ and $g(z)$ satisfies the CR equations.

Test the product:

$$f(z)g(z) = (u + iv)(s + it)$$

$$= (us - vt) + i(vs + ut)$$

$$\partial_x (us - vt) = \partial_x u + \partial_x s - \partial_x v - \partial_x t$$

$$\partial_x(vs + ut) = \partial_x v + \partial_x s + \partial_x u + \partial_x t$$

$$\partial_y(us - vt) = \partial_y u + \partial_y s - \partial_y v - \partial_y t$$

$$\partial_y(vs + ut) = \partial_y v + \partial_y s + \partial_y u + \partial_y t$$

Write the CR equations. These are not assumed to be true. They are to be tested with substitutions from (1,2,3,4). If the equalities hold after the substitutions, then we conclude that the sum satisfies the CR equations.

$$\partial_x(us - vt) = \partial_y(vs + ut)$$

$$\partial_x(vs + ut) = -\partial_y(us - vt)$$

Expand the CR equations:

$$\partial_x u + \partial_x s - \partial_x v - \partial_x t = \partial_y v + \partial_y s + \partial_y u + \partial_y t$$

$$\partial_x v + \partial_x s + \partial_x u + \partial_x t = -\partial_y u - \partial_y s + \partial_y v + \partial_y t$$

Substitute on the LHS from (1,2,3,4):

$$\partial_y v + \partial_y t + \partial_y u + \partial_y s = \partial_y v + \partial_y s + \partial_y u + \partial_y t \text{ satisfies CR 1}$$

$$-\partial_y u + \partial_y t + \partial_y v - \partial_y s = -\partial_y u - \partial_y s + \partial_y v + \partial_y t \text{ satisfies CR 2}$$

The product $f(z)g(z)$ satisfies the CR equations.