Neeham, Chapter 4, Exercise 10, p. 212.
G. Palmer, October 28, 2015

10. Instead of writing a mapping in terms of its real and imaginary parts (i.e. f = u + iv), it is sometimes more convenient to write it in terms of length and angle:

$$\mathbf{f(z) = Re^{i\psi},}$$

where R and ψ are functions of z. Show that the equations that chaaracterize an analytic f are now

$$\partial_x R = R\,\partial_y \psi \quad \text{and} \quad \partial_y R = -R\,\partial_x \psi$$

Differentiate with Leibnitz method:

$$\partial_x f = \partial_x R\, e^{i\psi} + iRe^{i\psi}\partial_x \psi \qquad (1)$$

$$\partial_y f = \partial_y R\, e^{i\psi} + iRe^{i\psi}\partial_y \psi \qquad (2)$$

Note that (1) and (2) have the same format as (7) and (8), p. 210.

Write Jacobian:

$$J = \begin{pmatrix} \partial_x R\, e^{i\psi} & Re^{i\psi}\,\partial_x \psi \\ \partial_y R\, e^{i\psi} & Re^{i\psi}\,\partial_y \psi \end{pmatrix}$$

$$= e^{i\psi}\begin{pmatrix} \partial_x R & R\,\partial_x \psi \\ \partial_y R & R\,\partial_y \psi \end{pmatrix}$$

The $e^{i\psi}$ cancel out when the partials are set equal. For J to reduce to an amplitwist, the matrix must have the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$(from $p.$ 210); then write the CR equations:

$$\partial_x R = R\,\partial_y \psi$$

$$\partial_y R = -R\,\partial_x \psi$$