Needham, Chapter 4, Exercises 1-3, p. 211
G. Palmer, October 12, 2015

1. Use the Cauchy-Riemann equation to verify that z ↦ $\overline{z}$ is not analytic.

Let z = x + iy

$\overline{z}$ = x - iy, u = x, v = -y

$\frac{\partial}{\partial x}(u) = 1$, $\frac{\partial}{\partial x}(v) = 0$

$\frac{\partial}{\partial y}(u) = 0$, $\frac{\partial}{\partial y}(v) = -1$

Test

$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial y}(v)$, 1 = -1, contradiction

$\frac{\partial}{\partial x}(v) = -\frac{\partial}{\partial y}(u)$, 0 = -0

The first C-R test fails. z ↦ $\overline{z}$ is not analytic.

2. The mapping z ↦ z^3 acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by Pi, and its linear dimensions expanded by 12. Where was the shape originally located. [There are two possibilities.]

Rewrite

$z = |z|\, e^{i\theta}$, $z^2 = |z|^2 e^{i2\theta}$, $z^3 = |z|^3 e^{i3\theta}$

The amplitwist described in the question can be written:

$12\, e^{i\mathrm{Pi}}$

We can write an equation for $f'(z^3)$ if z^3 is analytic. We test with the Cauchy-Riemann equations. We move this to the end to make the answer easier to follow.*

Take the first derivative in the usual fashion to obtain $3\,z^2$.

$f'(z) = 12\, e^{i\mathrm{Pi}} = 3\,z^2 = 3\;|z|^2\, e^{i2\theta}$

$12 = 3\,|z|^2$

$|z|^2 = 4$, $|z| = 2$

$\mathrm{Pi} = 2\theta \implies \theta = \pm\mathrm{Pi}/2$

$$z = 2\, e^{i(\pm \mathrm{Pi}/2)} = \pm 2i$$

The shape was originally located at ±2i.

*Footnote: The Cauchy-Riemann test on $z \mapsto z^3$

$$z^3 = \left(x^2 + 2\,xyi - y^2\right)(x + yi)$$

$$= x^3 + 2\,x^2\,yi - xy^2 + x^2\,yi - 2\,xy^2 - y^3\,i$$

$$= x^3 + 3\,x^2\,yi - 3\,xy^2 - y^3\,i$$

$$= \left(x^3 - 3\,xy^2\right) + \left(3\,x^2\,y - y^3\right) i$$

$$u = x^3 - 3\,xy^2, \quad v = 3\,x^2\,y - y^3$$

$$\frac{\partial}{\partial x}(u) = 3\,x^2 - 3\,y^2, \ \frac{\partial}{\partial x}(\mathrm{v}) = 6\,xy$$

$$\frac{\partial}{\partial y}(u) = -6xy, \ \frac{\partial}{\partial y}(v) = 3\,x^2 - 3\,y^2$$

Test

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial y}(v), \quad 3\,x^2 - 3\,y^2 = 3\,x^2 - 3\,y^2$$

$$\frac{\partial}{\partial x}(\mathrm{v}) = -\frac{\partial}{\partial y}(u), \quad 6\,xy = -(-6xy)$$

The C-R test confirms that $z \mapsto z^3$ is analytic.

3. Consider $z \mapsto \Omega(z) = \bar{z}^2/z$. By writing z in polar form, find out the geometric effect of Ω. Using two colours, draw two very small arrows of equal length emanating from a typical point z: one parallel to z; the other perpendicular to z. Draw their images emanating from Ω(z). Deduce that Ω fails to produce an amplitwist. [Your picture should show this in two ways.]

$$z = r\,e^{i\theta}, \quad \bar{z} = r\,e^{-i\theta}$$

$$\Omega(z) = \bar{z}^2/z = r^2\,e^{-i2\theta}/\,r\,e^{i\theta} = r\,e^{-i3\theta}$$

First perform the conjugation.

$$r\,e^{i\theta} \mapsto r\,e^{-i\theta}$$

When the small arrows are reflected around the real line, their angles are reversed, showing that the conjugation is non-conformal. The expansion is unitary.

Square the conjugate

$$r\, e^{-i\theta} \mapsto r^2\, e^{-2\,\theta}$$

When the $\bar{z}$ is squared, there is an expansion of r^2 and a twist of -θ = Arg $\bar{z}$. We were unsure whether we should square the lengths of the infinitesimal vectors, which would result in a contraction, or to apply an expansion of $2|v_{\bar{z}}|$. We applied the expansion, treating $2|v_{\bar{z}}|$ as a pseudo derivative.

$$v_{\bar{z}^2} = 2\, |v_{\bar{z}}|\ \text{Exp}[-i\theta]\ v_{\bar{z}}$$

Divide by z and apply the contraction and twist of -θ to the ray and all the arrows.

$$r^2 e^{-2\,\theta}/\text{r}\ e^{i\theta} = \text{r}e^{-3\,\theta}$$

The only way that I can see that Figure 1 demonstrates the lack of an amplitwist is in the reversal of the angles. The sizes of the two vectors appear to remain equal and the angle between them remains constant. Had we squared the infinitesimal vectors, both the angles and the lengths would have differed between the two vectors. Perhaps that is what was intended.

Out[2899]=

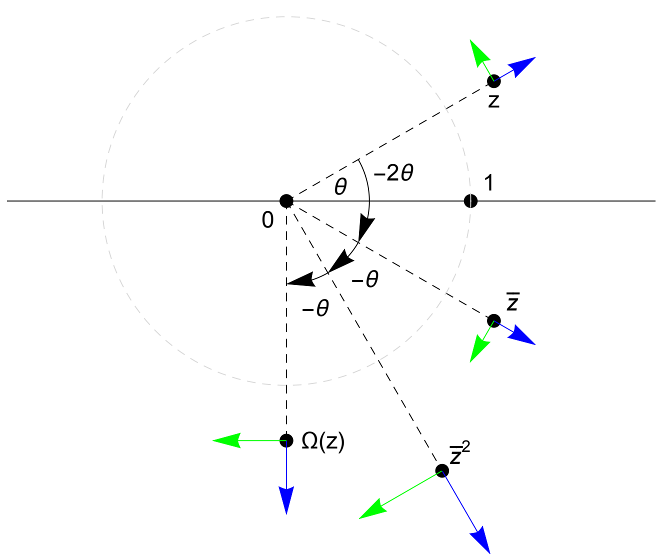

Figure 1. Local geometric effect of z ↦ $\bar{z}^2$/z

Source code for Figure 1 available on request.

```
(* Cauchy-Riemann test of z̄²/z *)

Clear["Global`*"]

z = x + y I;

Print["z̄² / z = ", exp = ((Conjugate[z]^2 / z) // ComplexExpand)];
```

$$u = \frac{x^3}{(x^2+y^2)^3} - \frac{3\,x\,y^2}{(x^2+y^2)^3};$$

$$v = -\frac{3\,x^2\,y}{(x^2+y^2)^3} + \frac{y^3}{(x^2+y^2)^3};$$

```
Print["Cauchy-Riemann test fails, ∂/∂x(u) ≠ ∂/∂y(v):\n", D[u, x], " ≠ ", D[v, y]]
```

$$\bar{z}^2 / z = \frac{x^3}{x^2+y^2} - \frac{3\,x\,y^2}{x^2+y^2} + i\left(-\frac{3\,x^2\,y}{x^2+y^2} + \frac{y^3}{x^2+y^2}\right)$$

Cauchy-Riemann test fails, $\frac{\partial}{\partial x}(u) \neq \frac{\partial}{\partial y}(v)$:

$$-\frac{6\,x^4}{(x^2+y^2)^4} + \frac{18\,x^2\,y^2}{(x^2+y^2)^4} + \frac{3\,x^2}{(x^2+y^2)^3} - \frac{3\,y^2}{(x^2+y^2)^3} \neq \frac{18\,x^2\,y^2}{(x^2+y^2)^4} - \frac{6\,y^4}{(x^2+y^2)^4} - \frac{3\,x^2}{(x^2+y^2)^3} + \frac{3\,y^2}{(x^2+y^2)^3}$$