Needham, Chapter 3, Notes

G. Palmer, October 7, 2022, 5:30 pm PDT

P. 124, Decomposition of Mobius transformation

Write the decomposition as a series of Mathematica™ functions which demonstrate the generic Möbius transformation $\frac{az+b}{cz+d}$.

$$f_4 \circ f_3 \circ f_2 \circ f_1(z) = \frac{a\,z + b}{c\,z + d}$$

Out[∘]=

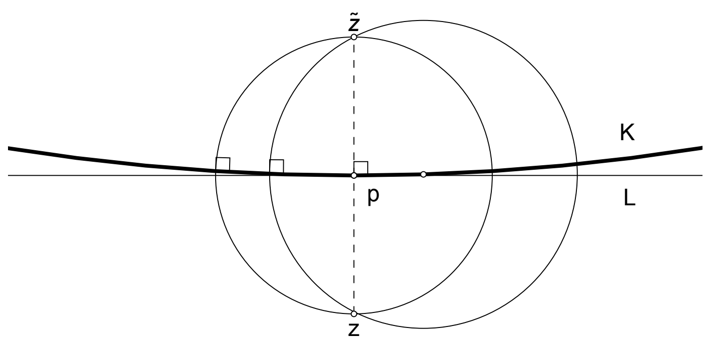


Figure 1. R of K tends to ∞

P. 130. Let us therefore choose L to be the real axis, and let p be the origin. The circle K of radius R centred at q = iR is therefore tangent to L at p. Using (4), we obtain [exercise].

The construction is represented in Figure 1. Use the second form of (4).

$$\mathcal{I}(z) = \frac{q\,\overline{z}+\left(R^2-|q|^2\right)}{\overline{z}-\overline{q}} = \frac{-i\left[\mathrm{iR}\,\overline{z}+\left(R^2-|\mathrm{iR}|^2\right)\right]/R}{-i\left(\overline{z}+\mathrm{iR}\right)/R} = \frac{\overline{z}-\mathrm{iR}+\mathrm{iR}}{-i\,\overline{z}/R+1} = \frac{\overline{z}}{1-i\,\overline{z}/R}$$

Out[•]=

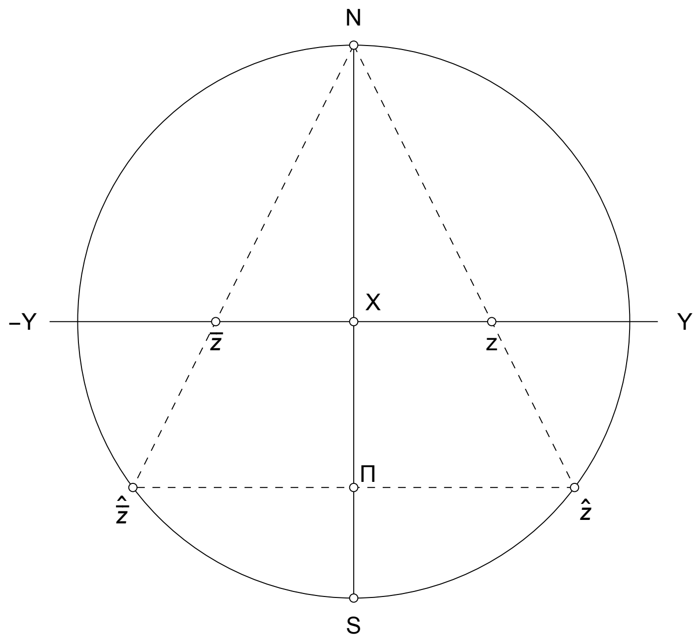


Figure 2. Complex conjugation in ℂ onto Riemann sphere

P. 143 *For example, what happens if we transfer* $f(z) = \bar{z}$ *to* Σ*. Clearly [exercise],*

Complex conjugation in ℂ *induces a reflection of the Riemann sphere in the vertical plane passing through the real axis.*

With the aid of Figure 2 one can demonstrate this proposition without the aid of algebra. Complex conjugation reflects z across the x axis, which is here labeled “X” because it also belongs to the Riemann sphere. The view is along the X axis. The figure shows a slice of the sphere on the vertical plane through the Y axis. The triangle $N\hat{z}\hat{\hat{z}}$ is isosceles by the definition of complex conjugation, so the mid-point of the base is a the plane Π seen on edge. Then $\hat{\hat{z}}$ is the reflection of $\hat{z}$ in Π, showing that *Complex conjugation in* ℂ *induces a reflection of the Riemann sphere in the vertical plane passing through the real axis.* To generalize this from the YZ plane to the whole sphere, one can imagine the triangle tilting on a hinge at N as z takes different values of x. The triangle expands and shrinks in width as z takes different values of y.

Out[◦]=

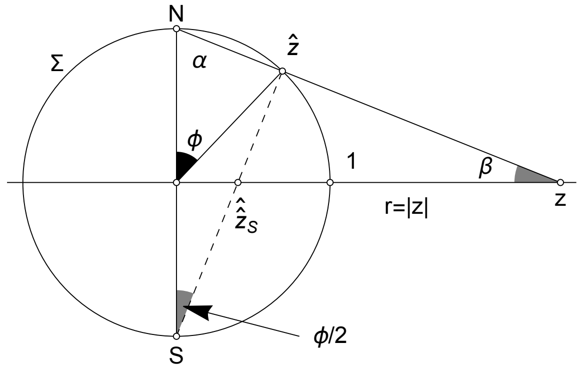


Figure 2. Cf. [23], p. 147

P. 147 *From [23b] it is clear [exercise] that the triangles N $\hat{z}$ S and N0z are similar....*

The triangles share the angle at N. They have equal angles at S and z. They both have right triangles by construction. Hence, they are similar.

P. 147 *It follows [exercise] that r = cot(ϕ/2).*

$$\sin(\phi/2) = \frac{1}{|N-z|}$$
$$\cos(\phi/2) = \frac{z}{|N-z|}$$
$$\cot(\phi/2) = \frac{z}{|N-z|} \cdot \left| N-z \right| = z$$

P. 150. *Solving the* [...] *equation* [w ↦ z = $M^{-1}(w)$] *for z in terms of w, we find [exercise] that* M^{-1} *is also a Möbius transformation:*

$$w = \frac{az+b}{cz+d}$$

$$w(cz + d) = az + b$$

$$wcz - az = b - wd$$

$$(cw - a)\, z = b - dw$$ (extract coefficients)

$$z = \frac{b-dw}{cw-a} = \frac{dw-b}{-cw+a}$$ (multiply top and bottom by –1)

$$M^{-1}(z) = \frac{dz-b}{-cz+a}$$ (swap z for w)

P. 151. *A simple calculation [exercise] shows that M is also a Möbius transformation.*

Instead of the subscripted coefficients, we use a, b, c, d, e, f, g, h.

$$(M_1 \circ M_2)(z) = \frac{\text{ez}+f}{\text{gz}+h}\left(\frac{\text{az}+b}{\text{cz}+d}\right)$$

$$= \frac{e\,\frac{\text{az}+b}{\text{cz}+d}+f}{g\,\frac{\text{az}+b}{\text{cz}+d}+h} = \frac{\text{aez}+\text{be}+\text{fcz}+\text{fd}}{\text{agz}+\text{bg}+\text{hcz}+\text{hd}}$$

$$= \frac{\text{aez}+\text{fcz}+\text{be}+\text{fd}}{\text{agz}+\text{hcz}+\text{bg}+\text{hd}} = \frac{(\text{ae}+\text{fc})\,z+(\text{be}+\text{fd})}{(\text{ag}+\text{hc})\,z+(\text{bg}+\text{hd})}$$

This is equivalent to (26).

P. 152. *If M(z) is normalized, then the two fixed points ξ_+, ξ_- are given by [exercise].*

Don’t be tempted to apply the normalization formula at this point. It will only lead to unwanted complications.

$$z = \frac{\text{az}+b}{\text{cz}+d}$$

$$z(\text{cz}+d) = \text{az}+b$$

$$\text{cz}^2 + \text{dz} - \text{az} - b = 0$$

$$z^2 + \left(\frac{d-a}{c}\right)z - \frac{b}{c} = 0$$

Apply the quadratic equation.

$$z = \xi_\pm = \frac{\frac{a-d}{c} \pm \sqrt{\left(\frac{d-a}{c}\right)^2 + 4\,\frac{b}{c}}}{2} = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4\,\text{bc}}}{2\,c}$$

Add and subtract 4ad under the root symbol.

$$= \frac{(a-d) \pm \sqrt{\left(d^2 - 2\,\text{ad} + 4\,\text{ad} + a^2\right) + 4\,\text{bc} - 4\,\text{ad}}}{2\,c}$$

Now apply (ad-bc) = 1 from p. 150.

$$\xi_\pm = \frac{(a-d) \pm \sqrt{(d+a)^2 + 4\,(\text{bc}-\text{ad})}}{2\,c} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2\,c}$$

This is what we wanted.

P. 160. *Using the fact that [M] is normalized, we find [exercise] that the characteristic equation is* $\lambda^2 - (a + d)\lambda + 1 = 0$.

Let M = ($\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right) = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

Substitute 1 for (ad – bc) because [M] is normalized.

p. 160.

tr[NP] = tr[PN]. (35)

this is easily verified by a direct calculation [exercise].

Let $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $P = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$N.P = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

tr[NP]= ae+bg + cf+dh

$$P.N = \begin{pmatrix} ea + fc & ga + hc \\ eb + fd & gb + hd \end{pmatrix}$$

tr[PN] = ea+fc + gb+hd = tr[NP] by rearranging

P. 169. Needham asserts *Since $(G \circ M) = (\tilde{M} \circ G)$, the so-called normal form of M is given by*

$$\frac{1}{w-\xi} = \frac{1}{z-\xi} + T.$$

The dervation from *(G ∘ M) = (M̃ ∘ G)* appears to come to us out of the blue, but all the information is available for a derivation of this and the following equation. Furthermore, it is easy to see that the equation is just

(parabolic M in w-plane) = (parabolic M in z-plane) + T

$$G \circ M = \frac{1}{z-\xi} \circ w = \frac{1}{w-\xi}$$

$$\tilde{M} \circ G = \tilde{M}(\tilde{z}) = \tilde{z} + T = G(z) + T = \frac{1}{z-\xi} + T$$

Hence $\frac{1}{w-\xi} = \frac{1}{z-\xi} + T.$

Since M maps z = ∞ to w = (a/c),

$$\frac{1}{(a/c)-\xi} = \frac{1}{\infty-\xi} + T = T \qquad (1)$$

From (27) we have

$$\xi_{\pm} = \frac{(a-d)\pm\sqrt{(a+d)^2-4}}{2c}$$

$$= \frac{(a-d)\pm\sqrt{4-4}}{2c} \text{ (substituting (a+d) = ± 2)}$$

$$= \frac{(a-d)}{2c}\text{, as given on p. 152.} \quad (2)$$

Returning to (1), multiply top and bottom by c and substitute (2) for ξ

$$T = \frac{1}{(a/c)-\xi} = \frac{1}{(a/c)-\frac{(a-d)}{2c}} = \frac{1}{\frac{a}{c}-\frac{a}{2c}+\frac{d}{2c}}$$

$$= \frac{c}{\frac{a+d}{2}} = \frac{c}{\frac{\pm 2}{2}} = \pm c \text{ (again (a+d) = ±2)}$$

where the "±" is the arbitrarily chosen sign of (a+d) (p. 169).

The paragraph following (42) caused some consternation when Needham wrote that $m = -1$, because $\frac{a-c}{a+c}$ is not generally −1, but then I realized that for 1/z, a = 0 and c = 1. Then m reduces to $\frac{-1}{1} = -1$.

P. 170. *Without bothering to solve for* m*, we now obtain [exercise] the following algebraic classification*: ...

elliptic, *iff (a+d) is real and* $|a+d| < 2$

This is a rotation which induces an equal rotation of Σ about the vertical axis through its centre and sends horizontal circles to themselves. Great circles through the poles are permuted among themselves. The archetypal form of the multiplier is $e^{i\alpha}$.

From (43), p. 170,

$$\sqrt{m} + \frac{1}{\sqrt{m}} = a + d \qquad (1)$$

From p. 152, elliptical transformations have form $e^{i\alpha} z$ with fixed points 0, ∞. From p. 164, §§2, $m = e^{i\alpha}$ if M(z) is an elliptic Möbius transformation. Substitute this m into (1).

$$e^{i\alpha/2} + e^{-i\alpha/2} = a + d$$

$$|e^{i\alpha/2} + e^{-i\alpha/2}| = 2\cos(\alpha/2)$$

Since $\alpha = \text{Arg}(m)$, the domain of α is $(-\pi \,..\, \pi]$. But where $\alpha = 0$, $2\cos(\alpha/2) = 2$, so we cannot allow this. Then stipulate $\alpha \in (-\pi \,..\, \pi] \mid \alpha \neq 0$. Then, $|a + d| < 2$, iff (a+d) real and $\alpha \neq 0$.

parabolic, *iff (a+d) = ±2*

The parabolic transformation has only one fixed point. From (28), p. 153: ∞ *is the sole fixed point if and only if M(z) is a translation, M(z) = z + b*. On p. 169, we find

If $M(z) = \frac{az+b}{cz+d}$ *is normalized, then we know from (27), p. 152, that it is parabolic if and only if (a+d) = ±2, in which case the single fixed point is* $\xi = (a-d)/2c$. *That is all we need to define this class.*

But is the fixed point ∞? M(z) = z + b can be rewritten as

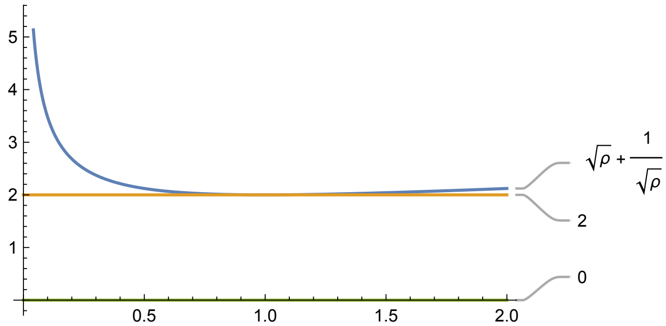
$$M(z) = \frac{z+b}{0 \cdot z + 1}$$

then a = 1, c = 0, and d = 1, so, keeping in mind that *(a+d) = ±2,*

$$\xi = \frac{(1-1) \pm \sqrt{(\pm 2)^2 - 4}}{2 \cdot 0} = \frac{0}{2 \cdot 0} = ?\infty$$

hyperbolic, *iff (a + d) is real and |a + d| > 2*

Out[ ]=



From p. 152, the induced transformation on the Riemann sphere takes the form of an origin-centred expansion $z \longrightarrow \rho z$ with fixed points 0, ∞. On p. 165 we find that $m = \rho \neq 1$. Then

$$\sqrt{m} + \frac{1}{\sqrt{m}} = \sqrt{\rho} + \frac{1}{\sqrt{\rho}} = |a + d|$$

because ρ is a positive real number by definition. Suppose $|a+d| \leq 2$. Then M would be an elliptical or a parabolic transformation. Since it cannot be either one of these, it must be the case that $|a + d| > 2$. Since $\sqrt{\rho} + \frac{1}{\sqrt{\rho}}$ are positive real numbers, they sum to $|a + d|$. Since $m \neq 1$, $| a + d | \neq 2$. Vasco pointed out that the plot of $\sqrt{\rho} + \frac{1}{\sqrt{\rho}}$ shows that $|a+d| > 2$.

loxodromic,   iff (a + d) is complex

The loxodromic transformation is both a rotation and an expansion. If $|a + d| < 2$ and $(a + d)$ real, M is elliptic. If $|a + d| = \pm 2$ and $(a + d)$ real, M is parabolic. If $|a + d| > 2$ and $(a + d)$ real, M is hyperbolic. These choices of $(a + d)$ account for all the real numbers. Then if M is loxodromic it must be complex. The multipliers are as follows: elliptics have rotations, parabolics have translations, and hyperbolics have expansions. The only remaining possibility is for m to include both an expansion and a rotation, so M must be loxodromic. In sum, if m of M has both a rotation and an expansion and (a+d) is complex, it must be loxodromic, and if it is loxodromic, it must have these qualities.

Vasco's answer is much more succinct.

Question: If one applied a parabolic M to a loxodromic M, would it be a parabolic loxodromic M?

P. 170. *Since* $\tilde{M}(z) = mz$*, it's normalized matrix is [exercise]*

$$[\tilde{M}] = \begin{bmatrix} \sqrt{m} & 0 \\ 0 & 1/\sqrt{m} \end{bmatrix}.$$

The matrix equivalent of $\tilde{M}(z) = \mathrm{m}z$ is

$$[\tilde{M}] = \begin{bmatrix} \mathrm{m} & 0 \\ 0 & 1 \end{bmatrix} \qquad (1)$$

But if $[\tilde{M}]$ represents $\tilde{M}(z)$, then so does k$[\tilde{M}]$, where k is a constant, either real or complex (pp. 159-160). By inspection, if we let $k = 1 \big/ \sqrt{\mathrm{m}}$ then

$$[\tilde{M}] \equiv \mathrm{k}[\tilde{M}] = \begin{bmatrix} \sqrt{\mathrm{m}} & 0 \\ 0 & 1 \big/ \sqrt{\mathrm{m}} \end{bmatrix}$$

This is what we wanted. But we must check to see that it is normalized. It is, because det(k $[\tilde{M}]$) = 1. We could multiply $[\tilde{M}]$ by k without changing the geometric nature of $\tilde{M}$ because $[\tilde{M}]$ in (1) is not normalized. But once $[\tilde{M}]$ is normalized, any further multiplication by a constant would imply a change in $\tilde{M}$ (p. 160).

One can accomplish the same thing through algebra, as Vasco did (p.c. Forum). Multiply (1) by the complex number β and solve for β by writing an equation for the determinant.

$$\beta[\tilde{M}] = \begin{bmatrix} \beta\mathrm{m} & 0 \\ 0 & \beta \end{bmatrix}$$

$$\det(\beta[\tilde{M}]) = \beta^2\,\mathrm{m} = 1$$

$$\beta = \sqrt{1/\mathrm{m}}$$

Substitute for β in $\beta[\tilde{M}]$.

$$[\tilde{M}] \equiv \beta[\tilde{M}] = \begin{bmatrix} \sqrt{\mathrm{m}} & 0 \\ 0 & 1 \big/ \sqrt{\mathrm{m}} \end{bmatrix}$$