Needham, Chapter 3, Exercise 9, p. 183
Gary B. Palmer, April 24, 2021, 8:30 am PDT

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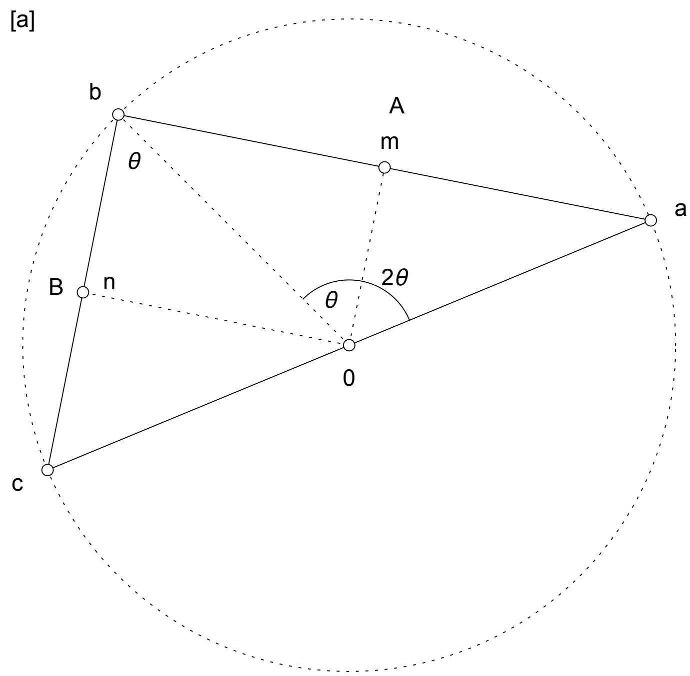

9 In order to analyze astronomical data, Ptolemy required accurate trigonometric tables, which he constructed using the addition formulae for sine and cosine. The figures below explain how he discovered these key addition formulae. Both the circles have unit radius.

(i) In figure [a], show that A = 2 sin θ and B = 2 cos θ.

As in [a], above, let b be the point where A and B intersect the unit circle. Let a and c be the second intersection points of A and B respectively. Let m and n be the midpoints of A and B. Then $\angle$ bm0 = $\pi/2$ by the construction of the isosceles triangle b0a, and $\angle$ m0b = θ. We can write

$$\sin\theta = A/2$$
$$A = 2\sin\theta$$

B is perpendicular to A by the construction of the inscribed triangle which has the diameter of the circle on one side, so it is also parallel to m. Triangle b0c is also isosceles, so $\angle$ bn0 = $\pi/2$. Then, since $\angle$bn0, $\angle$mbn, and $\angle$bm0 are all right angles, $\angle$m0n is also a right angle, so 0mbn is a rectangle. The diagonal of the rectangle creates similar triangles, so we can write

$$\cos\theta = B/2$$
$$B = 2\cos\theta$$

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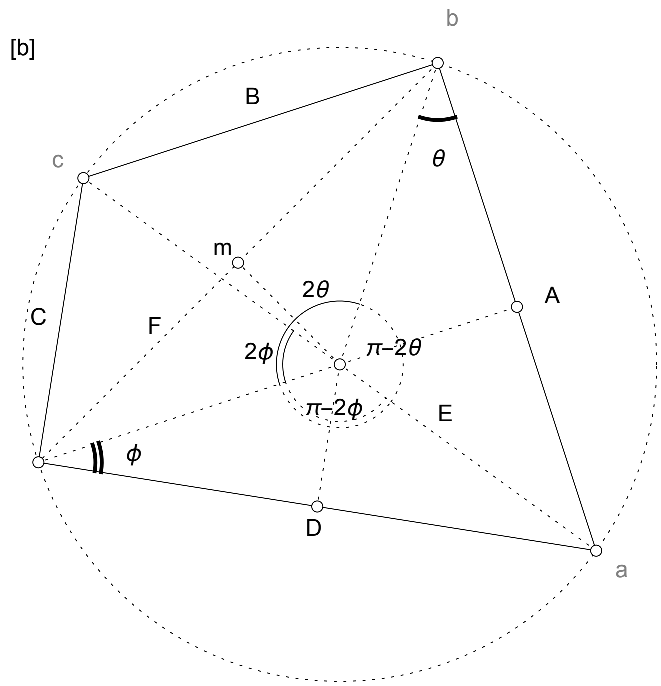

(ii) In figure [b], apply Ptolemy’s Theorem to the illustrated quadrilateral, and deduce that $\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$.

Since the illustrated quadrilateral appears to have a straight diagonal E from a to c, we add just the diagonal F. Then, by Ptolemy’s theorem AC + BD = E F.

$\sin\theta = B/2$
$B = 2\sin\theta$

$\sin\phi = C/2$
$C = 2\sin\phi$

Substitute for B and C:

$$AC + BD = E\,F \qquad (1)$$

$$A\,2\sin\phi + 2\sin\theta\,D = E\,F \qquad (2)$$

If we bisect F, we obtain two triangles each with angle at 0 of $(2\phi + 2\theta)/2 = (\phi + \theta)$. Then $F = 2\sin(\phi + \theta)$. Since E is a diagonal through the origin, E = 2. Now substitute into (2)

$$A\,2\sin\phi + 2\sin\theta\,D = 2 \bullet 2\sin(\phi + \theta)$$

$$A \sin \phi + \sin \theta \, D = 2 \sin(\phi + \theta) \qquad (3)$$

Now we need only show that $A = 2 \cos \theta$ and $D = 2 \cos \phi$. The angle subtended by A is π-2θ. We want $\angle$0ba, because the cosine of this angle equals A/2. Let α designate $\angle$0ba.

$2\,\alpha = \pi - (\pi - 2\theta) = 2\theta$
$\alpha = \theta$ (This is a surprise!)
$\cos \theta = A/2$
$A = 2 \cos \theta$

similarly, the angle subtended by D is $(\pi - 2\phi)$. Let β designate $\angle$0da.

$2\,\beta = \pi - (\pi - 2\phi) = 2\phi$
$\beta = \phi$
$\cos \phi = D/2$
$D = 2 \cos \phi$

Substituting in (3), we obtain

$$2 \cos \theta \sin \phi + \sin \theta \, (2 \cos \phi) = 2 \sin(\phi + \theta) \qquad (4)$$

Divide through by 2 and rearrange:

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \sin \phi \cos \theta$$

This is what we wanted.