Needham, Chapter 3, Exercise 8, p. 182
Gary B. Palmer, April 17, 2021, 9:20 pm PDT

Out[2814]=

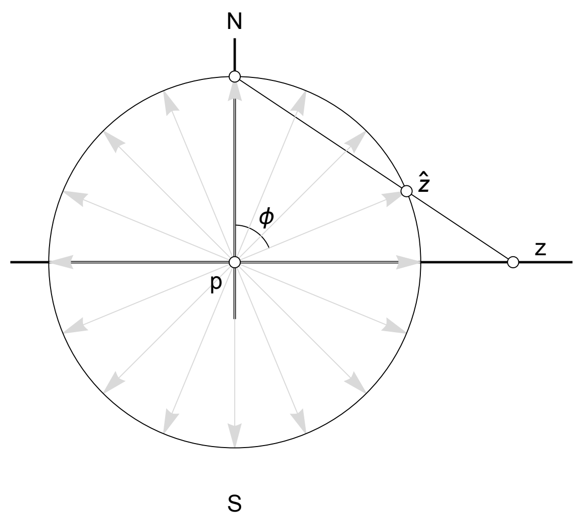


Figure 1. Rays to points on Riemann Sphere with projection to ℂ

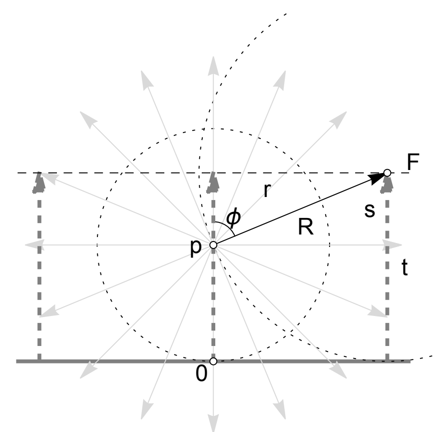


Figure 2. ℂ with v = 1 intersects light ray at F

*(8) This exercise continues the discussion of (2), p. 123. If a point p in space emits a flash of light, we claimed that each of the light rays could be represented by a complex number. Here is one, indirect method of establishing this correspondence. Once again, we choose units of space and time so that the speed of light is 1. After one unit of time, the expanding sphere of light emitted by p—made up of particles of light called photons—forms a unit sphere. Thus each photon may be identified with a point on the Riemann sphere, and hence, via stereographic projection, with a complex number. Indeed, if the photon has spherical polar coordinates ($\phi$, $\theta$), then (21)* [p. 147] *tells us that the corresponding complex number is* $z = \cot(\phi/2)\, e^{i\theta}$.

*Sir Roger Penrose (see Penrose and Rindler [1984, p. 13]) discovered the following remarkable method of passing from a light ray to the associated complex number directly, without the assistance of the Riemann sphere. Imagine that p is one unit vertically above the origin of the (horizontal) complex plane. At the instant that p emits its flash, let ℂ begin to travel straight up (in the direction* $\phi = 0$*) at the speed of light (=1) towards p. Decompose the velocity of the photon F emitted by p in the direction (*$\phi$*,* $\theta$*) into components perpendicular and parallel to ℂ. Hence find the time at which F hits ℂ. Deduce that F hits ℂ at the point* $z = \cot(\phi/2)\, e^{i\theta}$*. Amazingly, we see that Penrose's construction is equivalent to stereographic projection!*

We plot the construction on a slice (plane) passing through the verticle Z axis of a unit sphere Σ centered at p = {0, 0, 1}. The slice is taken at the angle $\theta$ from the X axis. Figure 1 shows angle $\phi$ of the ray of an illustrative photon. In Figure 1, z is calculated as $z = \cot(\phi/2)\, e^{i\theta}$.

*Figure 2 represents the construction described in paragraph 2. We assume that $0 \leq \phi \leq \pi$. The time at which F hits $\mathbb{C}$ is determined by $V_{\mathbb{C}} = 1$ and $V_{\mathbb{C}}\, t = t$, so the time equals the distance traveled by $\mathbb{C}$.*

*Hence find the time at which F hits $\mathbb{C}$: The time $t = 1 + s = 1 + \cos\phi/(1 - \cos\phi) = 1/(1 - \cos\phi)$. On $\mathbb{C}$, $z = r\, e^{i\theta}$.* (See derivations below.)

*Deduce that F hits $\mathbb{C}$ at the point $z = \cot(\phi/2)\, e^{i\theta}$: We* want to show that r = *$\cot(\phi/2)$*. The key feature of this construction is that R = t: The distance R traveled by the light ray in the direction *$(\phi, \theta)$* equals the vertical distance t traveled by $\mathbb{C}$. We decompose the velocity of the photon F emitted by p *in the direction $(\phi, \theta)$* into components $\cos\phi$ perpendicular to $\mathbb{C}$ and $\sin\phi$ parallel to $\mathbb{C}$. Then we obtain $s = R\cos\phi$, and eventually, $r = \cot\phi/2$.

| | | |
|---|---|---|
| $R^2 = r^2 + s^2$ | | (1) |
| $s = R\cos\phi$ | | (2) |
| $R = 1 + s \Longrightarrow s = R - 1$ | | |
| $R - 1 = R\cos\phi$ | | |
| $R - R\cos\phi = 1$ | | |
| $R = 1/(1 - \cos\phi)$ | | (3) |
| $s = \cos\phi/(1 - \cos\phi)$ | subst. (3) for R i n (2) | (4) |
| $r^2 = R^2 - s^2$ | rearrange (1) | |

$$r^2 = 1/(1 - \cos\phi)^2 - \cos^2\phi/(1 - \cos\phi)^2 \quad \text{subst. (3) for R and (4) for s}$$
$$= (1 - \cos^2\phi)/(1 - \cos\phi)^2$$
$$r = \sqrt{(1 - \cos^2\phi)/(1 - \cos(\phi))^2} = \sqrt{(1 - \cos^2\phi)}/(1 - \cos\phi)$$
$$= \sin\phi / (1 - \cos\phi)$$

Use substitutions $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = 1\text{-}2\sin^2\theta$.

$$r = 2\sin\phi/2\cos\phi/2 \; / \; (1 - (1 - 2\sin^2\phi/2))$$
$$= 2\sin\phi/2\cos\phi/2 \; / \; 2\sin^2\phi/2$$
$$= \cos\phi/2 / \sin\phi/2$$
$$= \cot\phi/2$$

Then it follows that *F hits $\mathbb{C}$ at the point $z = \cot(\phi/2)\, e^{i\theta}$*.