Needham, Ch 4, Ex. 7, p. 182
Gary Palmer, May 16, 2015, 12:25 pm

7. (i) Use a computer to draw the images in ℂ of several origin-centered circles under the exponential mapping, z -> e^z. Explain the obvious symmetry of these image curves with respect to the real axis.

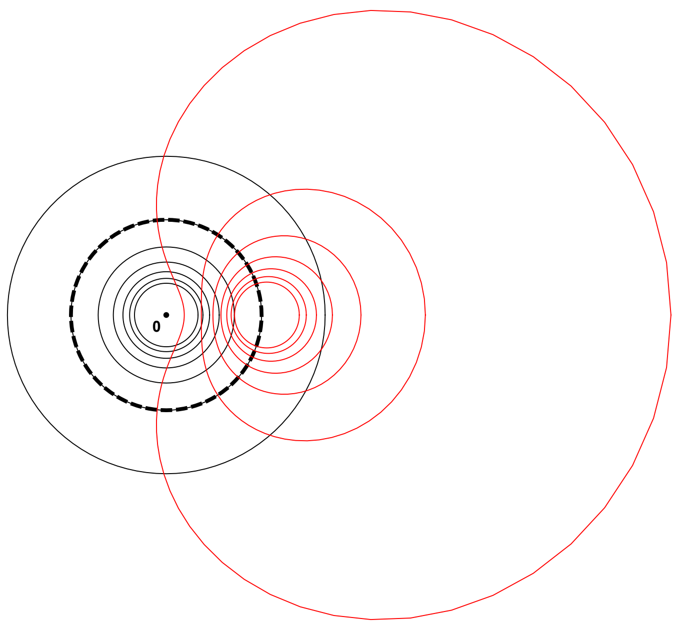

Figure 1. z ↦ e^z

See Figure 1. z can be written as x + yi. Then $e^z = e^{x+yi} = e^x e^{yi}$, where y is the angle of e^z in ℂ with respect to the real axis ($y = \text{Arg}(e^z)$), and e^x is a dilation of z. To explain the symmetry, if y is positive, then $\text{Arg}(e^z)$ is y radians counterclockwise, and if y is negative, then $\text{Arg}(e^z)$ is y radians clockwise. Since e^x is the same in both directions of rotation, the symmetry of the circles around the real axis in z is preserved in the image curves of e^z. Vasco has a neater explanation.

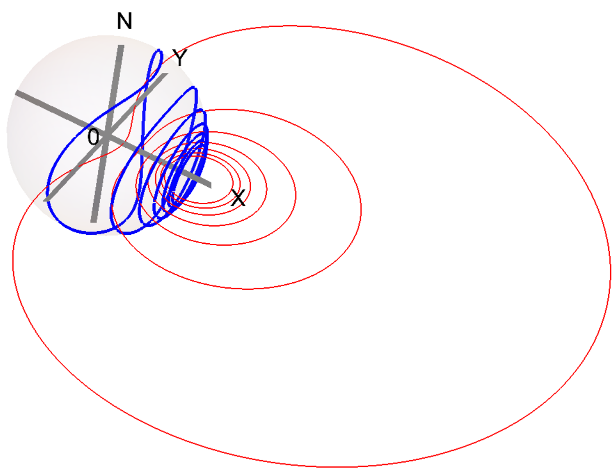

Figure 2a. Mapping $z \mapsto e^z$ in $\mathbb{C}$ (red)
projected to Riemann sphere (blue)

(ii) Now use the computer to draw these same image curves on the Riemann sphere instead of in $\mathbb{C}$. Note the surprising new symmetry!

The "new symmetry" to which Needham refers may be the symmetries across the XZ and XY planes, which can be seen more clearly in Figure 2b. Neither of these symmetries could exist in the $\mathbb{C}$ plane. One could also interpret this as a symmetry about the X (real) axis, an observation which may have implications for part (iii).

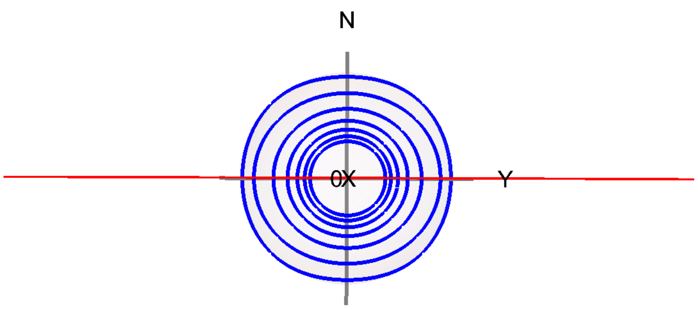

Figure 2b. Stereographic projection (blue) from Figure 2a viewed along X axis

(iii) Use (18) to explain this extra symmetry.

(18) The mapping of z ↦ (1/z) in ℂ induces a rotation of the Riemann sphere about the real axis through an angle of π.

The mapping *z ↦ (1/z)* applied to e^z induces a reflection in the real axis, making it appear that (1/z) is mapping the closed curves (red) to themselves. It follows from the graphical understanding of stereographic projection of the closed curves on the Riemann sphere, that the hemispheres on either side of the XZ plane reflect one another. We can show this with the equations for stereographic projection on p. 146.

$$\mathrm{X} + \mathrm{iY} = 2\,z/(1 + |\,z\,|^2) = 2\,\mathrm{r}e^{\mathrm{i}\theta}\big/(1 + r^2) = 2\,r(1 + r^2)\,e^{\mathrm{i}\theta}$$

$$= \frac{2\,r}{1+r^2}\,e^{\mathrm{i}\theta} = \frac{2\,r}{1+r^2}(\cos\theta + i\sin\theta)$$

$$Z = \frac{r^2-1}{r^2+1}$$

$$(X, Y, Z) = \left(\frac{2\,r}{1+r^2}\cos\theta,\ \frac{2\,r}{1+r^2}\sin\theta, \frac{r^2-1}{r^2+1}\right)$$

$$(\mathrm{X} + \mathrm{iY})_{1/z} = 2\,(1/r)\,e^{-\mathrm{i}\theta}\big/(1 + (1/r)^2) = [2\,(1/r)/(1 + r^2)]\,e^{-\mathrm{i}\theta}$$

$$= \frac{2}{r\left(1+(1/r)^2\right)} e^{-i\theta} = \frac{2\,re^{-i\theta}}{(r^2+1)} = \frac{2\,r}{(r^2+1)}\,(\cos(-\theta) + i\sin(-\theta))$$

$$(X, Y, Z)_{1/z} = \left(\frac{2\,r}{(r^2+1)}\cos(-\theta),\ \frac{2\,r}{(r^2+1)}\sin(-\theta),\ \frac{(1/r)^2-1}{(1/r)^2+1}\right)$$

$$= \left(\frac{2\,r}{(r^2+1)}\cos(-\theta),\ \frac{2\,r}{(r^2+1)}\sin(-\theta),\ \frac{1-r^2}{1+r^2}\right)$$

$$= \left(\frac{2\,r}{(r^2+1)}\cos(-\theta),\ \frac{2\,r}{(r^2+1)}\sin(-\theta),\ -\frac{r^2-1}{r^2+1}\right)$$

Now we can evaluate the motion in the Riemann sphere as induced by $z \longrightarrow 1/z$ in the real axis of $\mathbb{C}$.

motion in (X, Y, Z) = (none, across XZ plane, across XY plane)

There is no change in X with the projection of (1/z). The two reversals of sign in Y and Z constitute a rotation of π about the X axis. Hence, we expect to see symmetry in the XZ and XY planes when we view the projected curves from a vantage point on the x axis.