Needham, Chapter 3, Exercise 6, p. 182
Palmer, April 6, 2021, 1:35 pm PDT

Out[349]=

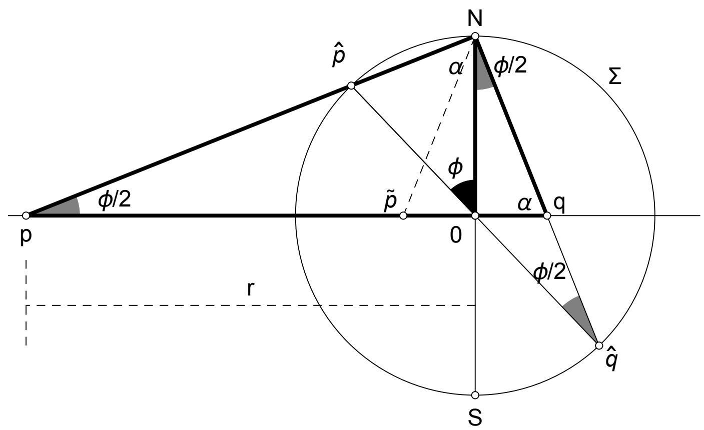

Figure 1. Cf. [a], p. 182

6. Both figures below show vertical cross sections of the Riemann sphere.

(i) In Figure [a], show that the triangles p0N and N0q are similar. Deduce (22).

(22) If p and q are antipodal points of Σ, then their stereographic projections p and q are related by the following formula:

$$q = -(1/\overline{p}).$$

$\angle \hat{p}\hat{q}N = \phi/2$ by the inscribed angle in a semicircle. Angle $0N\hat{q} = \phi/2$ *by equal angles at the base of an isosceles triangle. Then, since both p0N and N0q are right triangles, the two angles labeled α must be equal, with* $\alpha = \pi/2 - \phi/2$ and the angle at p must also be ϕ*/2. Since p0N and N0q have three corresponding congruent angles, the two triangles must be similar.*

Since this is the Riemann sphere, it has radius 1. Reflecting N0q in the NS axis, we obtain a third similar triangle N0$\tilde{p}$, which satisfies r/1 = 1/[0$\tilde{p}$], so $\tilde{p}$ is an inversion of p in C (See [21b] on p. 142, and Exercise 5, this chapter). Then by construction, q = $-\tilde{p} = -(1/\overline{p})$, as we wished to show.

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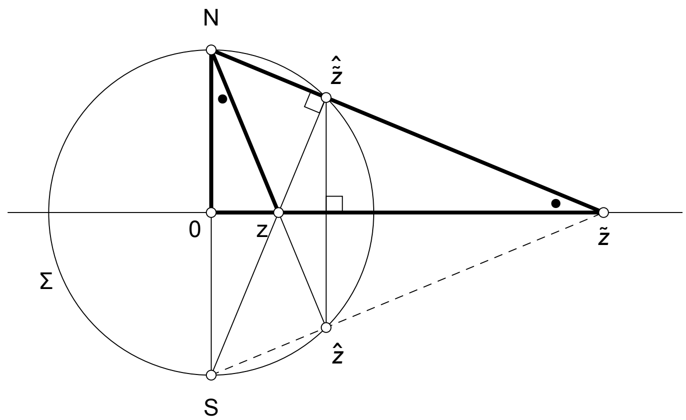

Figure 1[b]. Similar triangles z0N and N0$\tilde{z}$

(ii) Figure [b] is a modified copy of [21b]. Show that the triangles z0N and N0$\tilde{z}$ are similar. Deduce (17).

Both are right triangles. From Exercise 5 we know that $[0\,z]/1 = 1/\left[0\,\tilde{z}\right]$ implies $\tilde{z} = \mathcal{I}_C(z)$. Therefore, the angles with • are equal. Triangles z0N and N0$\tilde{z}$ have three corresponding angles in common, and they are therefore similar.

(17) Inversion of $\mathbb{C}$ in the unit circle induces a reflection of the Riemann sphere in its equatorial plane, $\mathbb{C}$.

The similar triangles shown in bold in Figure 2 are created by the projection from N to $\tilde{z}$ and from S to z from $\hat{\tilde{z}}$. We have shown that $\tilde{z}$ is the inverse of z. As shown in Exercise 5, since they are inverse, $\hat{z}$ is a projection point of z from N and $\hat{\tilde{z}}$ is a projection point of z from S. By the symmetry of triangle z $\hat{z}\hat{\tilde{z}}$ in the 0 $\tilde{z}$ axis, the triangle is symmetrical in $\mathbb{C}$, and therefore reflects $\hat{\tilde{z}}$ to $\hat{z}$ in $\mathbb{C}$, which establishes (17).