Needham, Chapter 3, Exercise 5, p. 181
Palmer, March 29, 2021, 1 pm PDT

Out[•]=

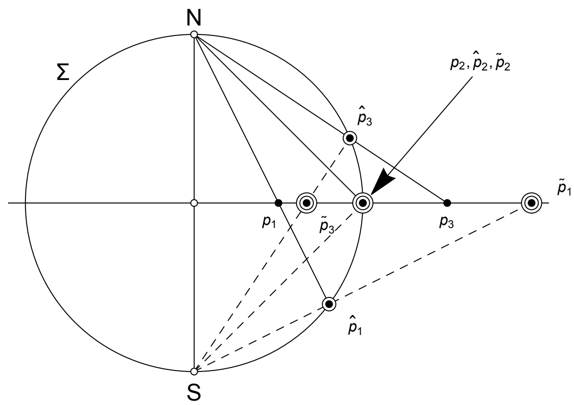

Figure 1. Projections $\mathbb{C}$ to Σ from N
and Σ to $\mathbb{C}$ from S

5. Consider the following two-stage mapping: first stereographically project $\mathbb{C}$ onto the Riemann sphere Σ in the usual way; now stereographically project Σ back to $\mathbb{C}$, but from the south pole instead of the north pole. The net effect of this is some complex mapping $z \mapsto f(z)$ of $\mathbb{C}$ to itself. What is f?

Figure 1 is a cross section of Σ formed by a plane Π passing through the NS axis. It traces the projections of three points in $\mathbb{C}$: p_1 inside the unit circle C, p_2 in C, and p_3 outside C. Points p_1 and p_3 appear as plain black points on the intersection of Π and $\mathbb{C}$; p_2, collinear with p_1 and p_3 is obscured by projections from N and S. The projections $\hat{p}_1$, $\hat{p}_2$, and $\hat{p}_3$ onto Σ of p_1, p_2, and p_3 from N are shown by a point centred within one circle. Since p_2 projects to itself at the equator, it is obscured by a symmetric projection from S. The projections $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}_3$ from Σ back to $\mathbb{C}$ are shown by a point centered in two concentric circles. Notice that the projections back to $\mathbb{C}$ from Σ do not all return to the original p_1, p_2, and p_3. However, on C, $\tilde{p}_2 = \hat{p}_2 = p_2$. Figure 1 shows that $\mathbb{C}$ outside C projects from N back to the northern hemisphere of Σ. $\mathbb{C}$ inside C projects from N to the southern hemisphere of Σ. From S, the southern hemisphere projects back to $\mathbb{C}$ outside C, but not to the initial points. From S, the northern hemisphere projects back to $\mathbb{C}$ inside C, but not to the initial points. C projects to itself from either hemisphere.

Out[ ]= 2.5

Out[ ]=

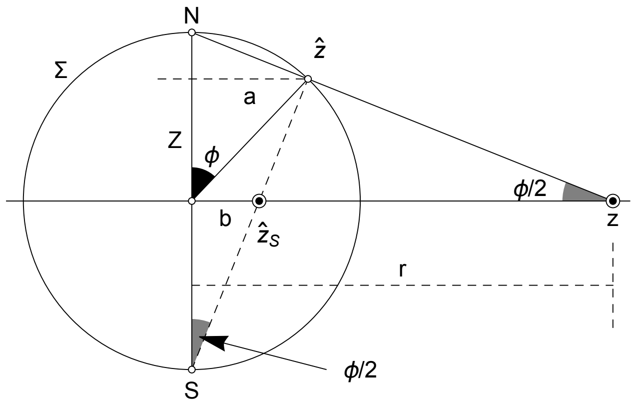

Figure 2. Cf. [23], p. 147

The function f(z) makes use of the Penrose approach described on p. 147.

$z = \cot(\phi/2)\, e^{i\theta}$

$\cot(\phi/2) = z e^{-i\theta}$

$\phi/2 = \operatorname{arccot}\left(z e^{-i\theta}\right)$ (1)

$\phi = 2 \operatorname{arccot}\left(z e^{-i\theta}\right)$ (2)

We want to find the projection of $\hat{z}$ from S onto $\mathbb{C}$. In Figure 2, this projection is labeled with $\hat{z}_S$. It has the same direction as z. One can write

$Z = \cos(\phi)$

$a = \sin(\phi)$

$b/a = 1/(1 + \cos(\phi))$

$b = \sin(\phi) / (1+\cos(\phi))$

$f(z) = \hat{z}_S = b e^{i\theta} = [\sin(\phi) / (1+\cos(\phi))] e^{i\theta}$

Since the only data point is z, it is necessary to substitute for ϕ with (1). Rearranging $e^{i\theta}$ gives

$f(z) = \hat{z}_S = b e^{i\theta} = \sin[2 \operatorname{arccot}\left(z e^{-i\,\mathrm{Im}(z)}\right)] e^{i\theta} / (1+\cos[2 \operatorname{arccot}\left(z e^{-i\theta}\right)])$

This obvious solution works, but it is a bit clumsy. The following more efficient solution also works.

$r = \cot(\phi/2)$ See notes, Ch 3, p. 147.

$b/1 = 1/r$ by similar triangles $0S\hat{z}_S$ and 0zN (3)

$b = 1/r = \tan(\phi/2) = \tan(\operatorname{arccot}\left(z e^{-i\theta}\right))$ subst. $\phi/2$ from (1)

$$f(z) = \hat{z}_S = b e^{i\theta} = \tan(\operatorname{arccot}(z e^{-i\theta})) e^{i\theta}$$

But I agree with Vasco (p.c. Forum), that Needham was probably looking for a more geometric solution. What I have above in (3) can be written $b \cdot r = 1$ (See Figure 2). Vasco deduced from the text and figure [1a] on p. 124 that this equation signals inversion in C. Hence, the formula for a double projection (from N and then S) is simply the formula $f(z) = 1/\overline{z}$, which is a very nice surprise. Since z contains the θ angle, one does not multiply by $e^{i\theta}$.