Needham, Chapter 3, p. 131

G. Palmer, March 14, 2021, 3:50 pm, PST

Out[719]=

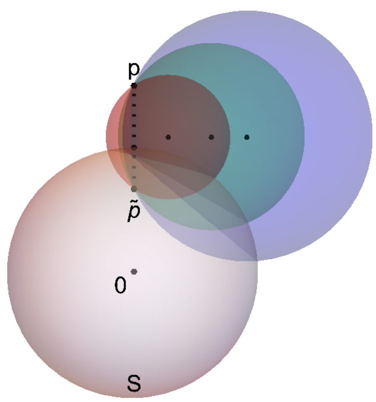


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Figure 1. Three spheres (R, G, B)
⊥ to S, centres aligned
viewed from x = 0, y = -2, z = 0
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Out[720]=

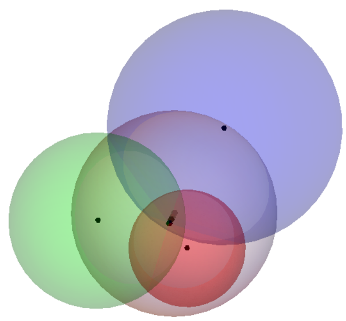

Figure 2. Three spheres in Figure 1
rotated about z axis by $-\pi/3$, $-\pi$, $-5\pi/3$
viewed from z = 2

Out[721]=

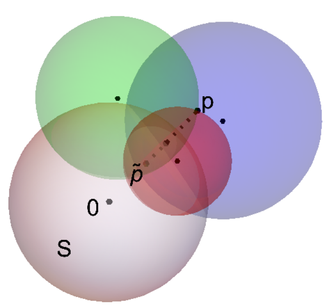

Figure 3. All spheres in Figure 2
rotated about origin from z axis
to {0.707107, -0.707107, 0.707107}
viewed from x = 0, y = -2, z = 0

3 Let S be a sphere, and let p be a point not on S. Explain why $I_s(p)$ may be constructed as the second intersection point of any three spheres that pass through p and are orthogonal to S. Explain the preservation of three dimensional symmetry in terms of this construction.

In two dimensions, from page 129, “The inverse $\tilde{z}$ of z in K is the second intersection point of any two circles that pass through z and are orthogonal to K.” From p. 135 we have that any two orthogonal spheres intersect a plane Π on which two orthogonal great circles pass through their centers. “Then S_1 and S_2 are orthogonal if and only if C_1 and C_2 are orthogonal.” Any sphere, say S_1, S_2, or S_3, that passes through p not on sphere S (say exterior to S) and is orthogonal to S has a great circle that is orthogonal to a great circle in S and has a second intersection point at $\tilde{p}$ in the interior to S. Since all three spheres S_1, S_2, and S_3, of the problem pass through p and are orthogonal to S, they all share a second intersection point $\tilde{p}$. Hence, $I_s(P)$ may be constructed as the second intersection point of any three spheres that pass through p and are orthogonal to S. This is the theory, but it seems to accord with the spirit of the question to demonstrate with plots of the construction.

Out[145]=

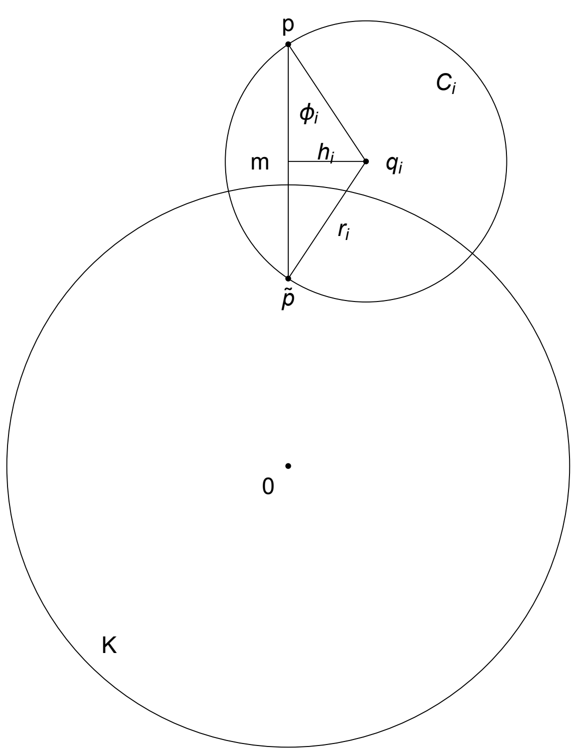

Figure 4. C ⊥ K

Construction of Figures 1 to 4.

Construct a circle K in the ℂ plane and pick a point p outside K (Figure 4). Use equation (4) to find $\tilde{p}$. Choose three real numbers r_1, r_2, and r_3 of convenient size for the radii of three circles, C_1, C_2, and C_3.

Calculate the midpoint m of p and $\tilde{p}$.

$$m = p - (p - \tilde{p})/2$$

Find the centre q_1 of C_1 as follows, where ϕ_1 is the angle between $(p-\tilde{p})$ and (q_1-p).

$$\phi_1 = \cos^{-1}(\mid p - m \mid / r_1)$$
$$h_1 = r_1 \sin(\phi_1)$$
$$q_1 = m + h_1$$

With q_1 and r_1, plot a circle that passes through p and $\tilde{p}$ and is orthogonal to K. Do the same for C_2, and C_3.

Convert the origin, p, $\tilde{p}$, q_1, q_2, and q_3 from complex points in $\mathbb{C}$ to points in (X, Y, Z) using (X, Y, Z) = (Re(z), 0, Im(z)). Rotate K and C_i from $\mathbb{C}$ to the XZ plane. Plot the spheres S and S_i, i = 1..3, for which K and C_i are great circles (Figure 1). The S_i are orthogonal to S and they each and all pass through the points p and $\tilde{p}$.

The S_i can be individually rotated about the axis passing through $(p-\tilde{p})$, which is the z axis in this case. The S_i remain orthogonal to S and continue to pass through $(p-\tilde{p})$ (Figure 2). The axis through $(p-\tilde{p})$ can be rotated about 0 to any direction in (X, Y, Z) (Figure 3). So long as the S_i undergo the same rotations about 0, they remain orthogonal to S and continue to pass through $(p-\tilde{p})$. The preservation of the relation to the points of inversion in S under these two rotations — (1) the rotations of S_i about the axis of the inverted points and (2) the directional rotation of all points about 0 — explains why $I_s(p)$ may be constructed as the second intersection point of any three spheres that pass through p and are orthogonal to S.

Explain the preservation of three dimensional symmetry in terms of this construction.

From (9), p. 128, we have

If a circle C does not pass through the centre q of K, then inversion in K maps C to another circle not passing through q.

From (13), we have

Let K be a plane or sphere, and let a and b be symmetric points with respect to K. Under a three-dimensional reflection in any plane or sphere, the images of a and b are again symmetric with respect to the image of K.

By the paragraph preceding this quote, "reflection" refers to inversion in a plane or sphere. I will regard a plane as a form of sphere. Explain the preservation of three dimensional symmetry in terms of our construction G (Figure 3) from orthogonal spheres and center planes on great circles. Plot a new sphere $S_{\mathcal{K}}$ so that G is external to $S_{\mathcal{K}}$. Then, by (13) the images s and $\tilde{s}$ of p and $\tilde{p}$ under reflection in $S_{\mathcal{K}}$ are again symmetric with respect to the image of S. By (9), any great circle on a center plane Π_i plane passing through both $S_{\mathcal{K}}$ and any sphere in G will map to another circle on Π_i in $S_{\mathcal{K}}$ not passing through $q_{\mathcal{K}}$. The inversion circles in the Π_i are great circles for inversion spheres in $S_{\mathcal{K}}$. All of the inversion spheres pass through s and $\tilde{s}$ and are orthogonal to $\tilde{S}$, the inversion of S in $S_{\mathcal{K}}$. Let $\tilde{G}$ be the construction G inverted in $S_{\mathcal{K}}$. The inversion of G to $\tilde{G}$ explains the presence of three dimensional symmetry.