Needham, Chapter 3, Exercise 27, p. 188
G. Palmer, October 31, 2021, 10:30 am PDT

27. Use (44), p. 170 to verify (53), p. 180.

elliptic:	*iff (a+d) is real and* $\lvert a+d \rvert < 2$;	*(44)*
parabolic:	*iff* $(a+d) = \pm 2$;	
hyperbolic:	*iff(a+d) is real and* $\lvert a+d \rvert > 2$;	
loxodromic:	*iff(a+d) is complex*	

If we define $\Phi = 2\cos^{-1} \mid \alpha \mid$, *then* M_α^ϕ *is* *(53)*
(i) elliptic if $|\phi| < \Phi$,
(ii) parabolic if $|\phi| = \Phi$
(iii) hyperbolic if $|\phi| > \Phi$

Verifications

I struggled with this exercise for days, trying to find a link between the algebra of the trace (**a**+**d**) and the algebra of the most general Möbius automorphism of the unit disc $M_\alpha^\phi(z) = e^{i\phi}\left(\frac{z-\alpha}{\overline{\alpha}\, z-1}\right)$. (Notice that the α in $M_\alpha^\phi(z)$ is not the same variable as the **a** in the trace). Along the way I also spent time wondering whether the three cases---elliptic, parabolic, and hyperbolic---could all be expressed with $M_\alpha^\phi(z)$. As it happens, they can. I consulted the Wikipedia page on Möbius transformations and found scant help there. I finally threw in the towel and consulted Vasco's published answer at https://sites.google.-com/site/vascoprat/needham. It was immediately evident that I had neglected the factor $e^{i\theta}$, which plays the key part in linking the trace to $M_\alpha^\phi(z)$. The exponential factor leads to a sine function, and another sine function is induced from the cosine function in φ, completing the link in an equation with two sines. (List (53) uses capital Φ, which in the Mathematica Greek font is hardly distinguishable from lower case ϕ, so here curly φ is substituted for capital Φ.)

Normalize $M_\alpha^\phi(z) = e^{i\phi}\left(\frac{z-\alpha}{\overline{\alpha}\, z-1}\right)$. The text does not specifically state that $M_\alpha^\phi(z)$ is normalized, but the list of criteria using the trace on p. 170 assumes normalization, so the coefficients of $M_\alpha^\phi(z)$ have to be normalized before the criteria can be applied. Multiply through by $e^{i\phi}$ and normalize, multiplying the coefficients by $1\Big/\sqrt{\mathbf{ad}-\mathbf{bc}}$, where **a**, **b**, **c**, and **d** are the coefficients of the general Möbius transformation interpreted in the context of $\frac{z-\alpha}{\overline{\alpha}\, z-1}$. All four coefficients are non-zero and $(\mathbf{ad} - \mathbf{bc}) \neq 0$. Let $A = \mid \alpha \mid^2$.

$$e^{i\phi}\left(\frac{z-\alpha}{\overline{\alpha}\, z-1}\right) = \left(\frac{e^{i\phi}\, z - e^{i\phi}\, a}{\overline{\alpha}\, z-1}\right)$$

Then, $\boldsymbol{a} = e^{i\phi}$, $\boldsymbol{b} = e^{i\phi}\,\alpha$, $\boldsymbol{c} = \overline{\alpha}$, $\boldsymbol{d} = -1$, $(\mathbf{ad} - \mathbf{bc}) = -e^{i\phi} - \left(-e^{i\phi}\,\alpha\,\overline{\alpha}\right) = e^{i\phi}(A-1)$.

$$\frac{1}{\sqrt{\mathbf{ad-bc}}} = 1\Big/\left(e^{i\phi/2}\sqrt{A-1}\right)$$

Now the new normalized coefficients **a** and **d** are $\boldsymbol{a} = e^{i\phi/2}\Big/\sqrt{A-1}$, $\boldsymbol{d} = -e^{-i\phi/2}\Big/\sqrt{A-1}$.

Find the trace: $(\boldsymbol{a}+\boldsymbol{d}) = e^{i\phi/2}\Big/\sqrt{A-1}$ - $e^{-i\phi/2}\Big/\sqrt{A-1} = 2\,i\sin(\phi/2)/i\,\sqrt{1-A} = 2\sin(\phi/2)\Big/\sqrt{1-A}$

$$|2\sin(\phi/2)|\Big/\sqrt{1-A} < 2 \Longrightarrow |\sin(\phi/2)|\Big/\sqrt{1-A} \qquad (1)$$

From (53),

$$\varphi = 2\cos^{-1}|\alpha|$$

$$\varphi/2 = \cos^{-1}|\alpha|$$

$$|\alpha| = \cos(\varphi/2) \Longrightarrow A = \cos^2(\varphi/2)$$

(i) The elliptic case:

$(\boldsymbol{a}+\boldsymbol{d})$ real and $|\boldsymbol{b}+\boldsymbol{d}| < 2$

$$|\boldsymbol{a}+\boldsymbol{d}| = \left|2\sin(\phi/2)\Big/\sqrt{1-\cos^2(\varphi/2)}\right| \qquad \text{[By substitution for A in (1)]}$$

$$= \left|2\sin(\phi/2)\Big/\sqrt{\sin^2(\varphi/2)}\right| = |2\sin(\phi/2)|\,/\,|\sin(\varphi/2)| \qquad (2)$$

$$|2\sin(\phi/2)|\,/\,|\sin(\varphi/2)| < 2$$

$$\Longrightarrow |\sin(\phi/2)| < |\sin(\varphi/2)|$$

$\varphi/2$ varies on $-\pi/2 \le \varphi/2 \le \pi/2$, because it's cosine is the positive real number, $|\alpha|$, which entails that $0 \le |\sin(\varphi/2)| \le 1$.
Since the elliptic case involves a complete rotation on the invariant circles, ϕ varies on the domain $-2\pi \le \phi \le 2\pi$, while $\phi/2$ varies on $-\pi \le \phi/2 \le \pi$, which entails that $0 \le |\sin(\phi/2)| \le 1$. Hence $|\sin(\phi/2)|$ and $|\sin(\varphi/2)|$ both vary on the range [0..1]. Then $|\sin(\phi/2)| < |\sin(\varphi/2)|$ entails that (44) validates (53), that $|\phi| < \varphi$.

(ii) The parabolic case: Here, the parabolic case is not the archetypal form described on p. 153, but a subcategory of $M_\alpha^\phi(z) = e^{i\phi}\left(\frac{z-\alpha}{\overline{\alpha}\,z-1}\right)$. It transforms parallel lines on $\mathbb{C}$ to invariant circles with a common

tangent at ∞ on the Riemann sphere (p. 153). ∞ is the only fixed point.

$$(\boldsymbol{a}+\boldsymbol{d}) = 2\sin(\phi/2)/\sin(\varphi/2) = \pm 2$$

$$= \sin(\phi/2)/\sin(\varphi/2) = \pm 1$$

$$\Longrightarrow \sin(\phi/2) = \pm\sin(\varphi/2)$$

Since $-\pi \leq \phi/2 \leq \pi$, $0 \leq |\sin(\phi/2)| \leq 1$. The angle $\varphi/2$ has the same domain $-\pi/2 \leq \varphi/2 \leq \pi/2$ as in the elliptic case, which entails that $-1 \leq \sin(\varphi/2) \leq 1$. Then we can write

$$|\sin(\phi/2)| = \sin(\varphi/2)$$

Then $(\boldsymbol{a}+\boldsymbol{d}) = \pm 2$ in (44) validates $|\phi| = \varphi$ in (53).

(iii) The hyperbolic case: A hyperbolic invariant curve is a line through the origin. A hyperbolic Möbius transformation induces expansion and contraction of $\mathbb{C}$. The archetypal hyperbolic Möbius transformation can be written $z \mapsto \rho z$. On the Riemann sphere, *The invariant curves are the great circles through the fixed points at the poles* (p. 152).

The proof proceeds exactly as in the elliptic case with only a switch in the direction of the inequality.

$$(\boldsymbol{a}+\boldsymbol{d}) \text{ real and } |\boldsymbol{a}+\boldsymbol{d}| > 2$$

$$|\boldsymbol{a}+\boldsymbol{d}| = |2\sin(\phi/2)| / |\sin(\varphi/2)| \qquad \text{(From (2))}$$

$$|2\sin(\phi/2)| / |\sin(\varphi/2)| > 2$$

$$\Longrightarrow |\sin(\phi/2)| > |\sin(\varphi/2)|$$

$\varphi/2$ varies on $-\pi/2 \leq \varphi/2 \leq \pi/2$, because it's cosine is the positive real number, $|\alpha|$, which entails that $0 \leq |\sin(\varphi/2)| \leq 1$.
ϕ varies on the domain $-2\pi \leq \phi \leq 2\pi$, while $\phi/2$ varies on $-\pi \leq \phi/2 \leq \pi$, which entails that $0 \leq |\sin(\phi/2)| \leq 1$. Hence $|\sin(\phi/2)|$ and $|\sin(\varphi/2)|$ both vary on the range [0..1]. Then $|\sin(\phi/2)| > |\sin(\varphi/2)|$ entails that (44) validates (53), that $|\phi| > \varphi$.