Needham, Chapter 3, Exercise 26, p. 188.
G. Palmer, October 17, 2021, 7:30 pm PDT

26. Reconsider (52), p. 178.

$$M_a(z) = \frac{z-a}{\bar{a}\,z-1} \qquad (52)$$

(i) Use (44), p. 170 to show that M_a is elliptic.

By (44), a <u>normalized</u> Möbius transformation is elliptic iff (a+d) is real and |a+d| < 2. Re. (52), a = 1, d = -1. Write M_a in matrix format:

$$[M_a] = \begin{bmatrix} 1 & -a \\ \bar{a} & -1 \end{bmatrix}$$

To normalize $[M_a]$, multiply by $\pm 1/\sqrt{\text{ad} - \text{bc}}$, from p. 150, substituting the coefficients of $[M_a]$ for a, b, c and d. Because the Mathematica .pdf renderer does not typeset the full expression properly, let $A = |\,a\,|^2$.

$$[M_a] = \frac{1}{\sqrt{1 \times (-1) - (-a)\,\bar{a}}} \begin{bmatrix} 1 & -a \\ \bar{a} & -1 \end{bmatrix} = \frac{1}{\sqrt{A-1}} \begin{bmatrix} 1 & -a \\ \bar{a} & -1 \end{bmatrix}$$

Using normalized values for a and d,

$$(a + d) = \frac{1}{\sqrt{A-1}}\,(1 - 1) = 0, \text{ which is real}$$

$$|\,a + d\,| = 0 < 2$$

Normalized M_a meets the criteria for an elliptic normalized Möbius transformation.

(ii) Use (43), p. 170 to show that both multipliers are given by m *= -1.*

$$(43)\ \sqrt{\text{m}} + \frac{1}{\sqrt{\text{m}}} = a + d$$

Substituting from part (i),

$$\sqrt{\text{m}} + \frac{1}{\sqrt{\text{m}}} = 0$$

$$\sqrt{\text{m}}\,\left(\sqrt{\text{m}} + \frac{1}{\sqrt{\text{m}}}\right) = 0$$

$$\text{m} + 1 = 0 \Longrightarrow \text{m} = -1$$

(iii) Calculate the matrix product `[Ma][Ma]`*, and thereby verify that* M_a *is involutory.*

If M_a is involutory, then

`[Ma][Ma]` = c[ε], where c is a complex constant.

Continue with normalized values from (i). Again, let $A = |a|^2$.

$$[M_a] = \frac{1}{\sqrt{A-1}} \begin{bmatrix} 1 & -a \\ \overline{a} & -1 \end{bmatrix}$$

$$[M_a][M_a] = \frac{1}{A-1} \begin{bmatrix} 1 & -a \\ \overline{a} & -1 \end{bmatrix}^2 = \frac{1}{A-1} \begin{bmatrix} 1-A & -a+a \\ \overline{a}-\overline{a} & 1-A \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -[\varepsilon], \text{ where c = -1}$$

This is what we wanted.

(iv) Use (27), p. 152 to calculate the fixed points of M_a*.*

The fixed points of normalized M_a are given as

$$\xi_{\pm} = \frac{1}{2c} \left((a-d) \pm \sqrt{\left((a+d)^2 - 4 \right)} \right) \qquad (27)$$

Make substitutions with normalized a, c, and d. Again, let $A = |a|^2$.

$$\xi_{\pm} = \frac{\left(\frac{1}{\sqrt{A-1}} - \frac{-1}{\sqrt{A-1}} \right) \sqrt{0-4}}{2 \frac{\overline{a}}{\sqrt{A-1}}} = \frac{\frac{2}{\sqrt{A-1}} \pm 2i}{2 \frac{\overline{a}}{\sqrt{A-1}}} = \frac{1}{\overline{a}} \left(1 \pm i \sqrt{A-1} \right) = \frac{1}{\overline{a}} \left(1 \pm \sqrt{1-A} \right) \quad \text{(i moved inside root)}$$

Since $M_a(z)$ is an automorphism of the unit disc D, $(1-A) \geq 0$ and both numerators are real.

(v) Show that the result of the previous part is in accord with figure [38].

Figure [38] shows that $\xi_- = \frac{1}{\overline{\xi_+}}$. We would like to show that $\xi_- \overline{\xi_+} = 1$. Use A = Abs[$a$]^2. Substitute into ξ_+ and ξ_- from part (iv), where the subscripts correspond to the signs on the roots,

$$\overline{\xi_+} = \frac{1}{a}\left(1 + \sqrt{1 - A}\right)$$

$$\xi_- \overline{\xi_+} = \frac{1}{\overline{a}}\left(1 - \sqrt{1 - A}\right) \frac{1}{a}\left(1 + \sqrt{1 - A}\right) = \frac{1}{A}\,(1 - (1 - A) = \frac{A}{A} = 1$$

This shows that the fixed points in part (iv) accord with the constraint $\xi_- = \frac{1}{\overline{\xi_+}}$ shown in figure [38].