Needham, Chapter 3, Exercise 25, p. 188.
G. Palmer, October 18, 2015, 10:20 am PDT

25. Let M(z) be the general Möbius automorphism of the upper half-plane.

(i) Observing that M maps the real axis into itself, use (29) to show that the coefficients of M are real.

$$\frac{(w-\tilde{q})(\tilde{r}-\tilde{s})}{(w-\tilde{s})(\tilde{r}-\tilde{q})} = [\mathrm{w}, \tilde{q}, \tilde{r}, \tilde{s}] = [\mathrm{z, q, r, s}] = \frac{(z-q)(r-s)}{(z-s)(r-q)} \qquad (29)$$

From subsection 6 The Cross Ratio, p. 154, it appears that the general automorphism of the UHP can be written

$$M = M^{-1}_{\tilde{q}\,\tilde{r}\,\tilde{s}} \circ M_{\mathrm{qrs}}$$

Equation (1) can be rewritten as (29), where [z,q,r,s] represents M_{qrs} and $[w,\tilde{q}, \tilde{r}, \tilde{s}]$ represents $M_{\tilde{q}\,\tilde{r}\,\tilde{s}}$. We can use (29) to solve for an M that maps the real line to the real line.

Choose $0,\ 1,\ \infty$ to be the three real numbers q', r', s' as in paragraph 4. Then, because M must map the real axis to itself, choose another three real numbers q, r, s and still another three real numbers $\tilde{q}, \tilde{r}, \tilde{s}$ again in the order corresponding to the order q, r, s. Substitute the chosen values of q, r, s and $\tilde{q}, \tilde{r}, \tilde{s}$ into (29). Now this restraint of (29) to real numbers represents an M that sends any three numbers q, r, s on the real line to any three other numbers $\tilde{q}, \tilde{r}, \tilde{s}$ on the real line. Using (1), expand M_{qrs} and $M^{-1}_{\tilde{q}\,\tilde{r}\,\tilde{s}}$ and write them in matrix form:

$$M_{\mathrm{qrs}} = \frac{(z-q)(r-s)}{(z-s)(r-q)} = \frac{(r-s)\,z-q(r-s)}{(r-q)\,z-s(r-q)} \equiv \begin{bmatrix} r-s & -q(r-s) \\ r-q & -s(r-q) \end{bmatrix} = \left[M_{\mathrm{qrs}}\right]$$

$$M_{\tilde{q}\,\tilde{r}\,\tilde{s}} = \frac{(\tilde{r}-\tilde{s})\,w-\tilde{q}(\tilde{r}-\tilde{s})}{(\tilde{r}-\tilde{q})\,w-\tilde{s}(\tilde{r}-\tilde{q})} \equiv \begin{bmatrix} \tilde{r}-\tilde{s} & \tilde{q}(\tilde{r}-\tilde{s}) \\ \tilde{r}-\tilde{q} & \tilde{s}(\tilde{r}-\tilde{q}) \end{bmatrix} = \left[M_{\tilde{q}\,\tilde{r}\,\tilde{s}}\right]$$

$$\left[M^{-1}_{\tilde{q}\,\tilde{r}\,\tilde{s}}\right] = \begin{bmatrix} \tilde{s}(\tilde{r}-\tilde{q}) & \tilde{q}(\tilde{r}-\tilde{s}) \\ -(\tilde{r}-\tilde{q}) & \tilde{r}-\tilde{s} \end{bmatrix}$$

All the entries in $\left[M_{\mathrm{qrs}}\right]$ and $\left[M^{-1}_{\tilde{q}\,\tilde{r}\,\tilde{s}}\right]$ are real numbers by the substitutions into (29) which reflect a mapping from the real line to the real line. Hence, all of the entries in $[M] = \left[M^{-1}_{\tilde{q}\,\tilde{r}\,\tilde{s}}\right]\left[M_{\mathrm{qrs}}\right]$ are real numbers. It follows that all the coefficients of M are real numbers.

(ii) By considering Im[M(i)], deduce that the only restriction on these real coefficients is that they have positive determinant (ad - bc) > 0.

We know that M cannot map i to the real line, because it maps the real line to itself. Since M is an automorphism of the UHP, it is established that Im[M(i)] > 0.

$$M(i) = \frac{ai + b}{ci + d}, \text{ where a, b, c, d are real, as shown in (i).}$$

In order to separate the real and imaginary parts of M(i), eliminate i in the denominator.

$$M(i) = \frac{(ai + b)(ci - d)}{(ci + d)(ci - d)} = \frac{-ac - adi + bci - bd}{-c^2 - cdi + cdi - d^2}$$

$$= \frac{(bc - ad)\,i - (ac + bd)}{-c^2 - d^2} = \frac{(ad - bc)\,i + (ac + bd)}{c^2 + d^2}$$

$$\text{Im}[M(i)] = \frac{(ad - bc)}{c^2 + d^2}$$

Then since we have already established that Im[M(i)] > 0, it follows that (ad-bc) > 0. So (ad — bc) > 0 if Im[M(i)] > 0. Suppose that (ad—bc) ≤ 0. Then

$$\frac{(ad - bc)}{c^2 + d^2} \leq 0$$

which means that Im[M(i)] $\not>$ 0 and M is not and automorphism of the UHP, which contradicts the premise of the exercise. Hence, (ad-bc) > 0 is the *only* restriction. There is no need for any restriction on the signs or magnitudes of the coefficients.

(iii) Explain (both algebraically and geometrically) why these Möbius transformations form a group under composition.

This is the group of all Möbius automorphisms of the UHP, which we have shown to have only real coefficients with determinant (ad-bc) > 0.

Algebraically:

1. Members of the group have an identity transformation ε which belongs to the group. That is, all the coefficients of ε are real and det ε > 0. We show in matrix format that [ε] [z] = [z] and [ε] [M] = [M].

$$[\varepsilon] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Since a composition of two members of the group must be a member of the group, we can apply [ε] to a matrix [M] of the group and determine whether the coefficients of the result are real, (ad — bc) > 0, and the result is equal to [M].

$$M = \frac{az+b}{cz+d} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where a, b, c, d are real and (ad — bc) > 0}$$

$$[\varepsilon][M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ 0+c & 0+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The result equals [M], so ε is the identity transformation. The identity transformation ε is a member of the group, because for ε itself, a, b, c, d are real and (ad — bc) > 0.

2. Members of the group have an inverse. From (ii), we know that det [M] > 0. One should find that $[M]^{-1}[M] = [\varepsilon]$.

$$[M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$[M]^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$[M]^{-1}[M] = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\varepsilon]$$

All the coefficients of $[M]^{-1}$ are real, $[M]^{-1}[M]$ results in the identity transformation, and det $[M]^{-1} > 0$.

3. The composition of two member Möbius transformations M ∘ N, is a member of the group.

$$[N][M] = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{bmatrix}$$

Since a..h are all real numbers, the four resultant components on the RHS are also real by the group property of real numbers under linear transformations and the definition of matrix multiplication. From p. 159, det $\{[M_2][M_1]\} = \det[M_2]\det[M_1]$. Since det [N] > 0 and det [M] > 0, det [N][M] > 0. We have therefore shown algebraically that the group of all Möbius transformations having only real coefficients with determinant (ad-bc) > 0 forms a group under composition.

Geometrically:

Geometrically, these are automorphisms of the upper half plane. Therefore both z and M(z) are in the upper half plane. We have shown algebraically that they have an identity and each one has an inverse. The identity ε on z in the upper half plane is in the upper half plane by the definition of z; M^{-1} is in the upper half plane because $M^{-1} \circ M(z) = \varepsilon(\mathrm{z})$, which is in the upper half plane; and the composition of two such automorphisms is in the upper half plane because the coefficients of the product are all real and det(M) > 0. Hence, they form a group.

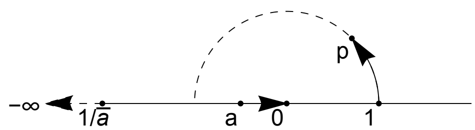

Figure 1. Möbius transformation of upper half−plane

(iv) How many degrees of freedom does M have? Why does this make sense?

We have the constraint that (ad - bc) > 0. The reasoning utilized by Needham (p. 176) in determining that automorphisms of D have three degrees of freedom appears to apply to automorphisms of the upper half-plane. To paraphrase, The set of all Möbius transformations forms a six parameter family. We lose three degrees of freedom when we impose the condition that M(z) map the UHP to itself, just as we lost three for M_a^ϕ for the unit disc. We attempt to find a formula by the symmetry principle. The discussion of M_a^ϕ appears to apply here. We can place a point a on the real line and arrange to send it to zero. Then $1/\bar{a}$ should be sent to ∞. But with the stipulation that all our coefficients are real, M_a^ϕ sends a to 0 and both $1/\bar{a}$ and 1/a to infinity (Figure 1). Because a represents only a single bit of information, we have lost another degree of freedom, leaving us with two. We can send points to any point on the real axis and then rotate them by $e^{i\phi}$. The domain of the rotation is limited to the upper half circle, [0, Pi]. We conclude that unlike M_a^ϕ for the unit disc, the Möbius automorphism of the UHP, M_a^ϕ, with real a, $0 \leq \phi \leq$ Pi, has only two degrees of freedom: one bit for a and one for ϕ.

Why does this make sense?

It makes sense because the Möbius transformations represents two reflections or inversions. For the mapping of the UHP to itself, the second inversion would be an inversion in the real line. This would map a to a, where a is a point on the real line. Hence, no new information is required.

I made an argument based on the Riemann mapping theorem which appears to contradict my first answer.

Any simply connected region R (other than the entire plane) may be mapped one-to-one and conformally to any other such region S. (p. 180)

We let R be the upper half-plane (UHP), which is simply connected. M is an automorphism from the UHP to the UHP, so we can let S also be the upper half plane. These regions can also be mapped one-to-one and conformally to and from D (p. 187, Ex. 24). Hence, the Riemann mapping theorem applies. Furthermore “the number of one-to-one and conformal mappings from R to S is equal to the number from R to D” (p. 180) and equal to the number of automorphisms of D. Therefore, “there exists a 3-parameter family of one-to-one mappings from R to S” and by construction from the UHP to the UHP. I believe a 3-parameter family implies 3 degrees of freedom.