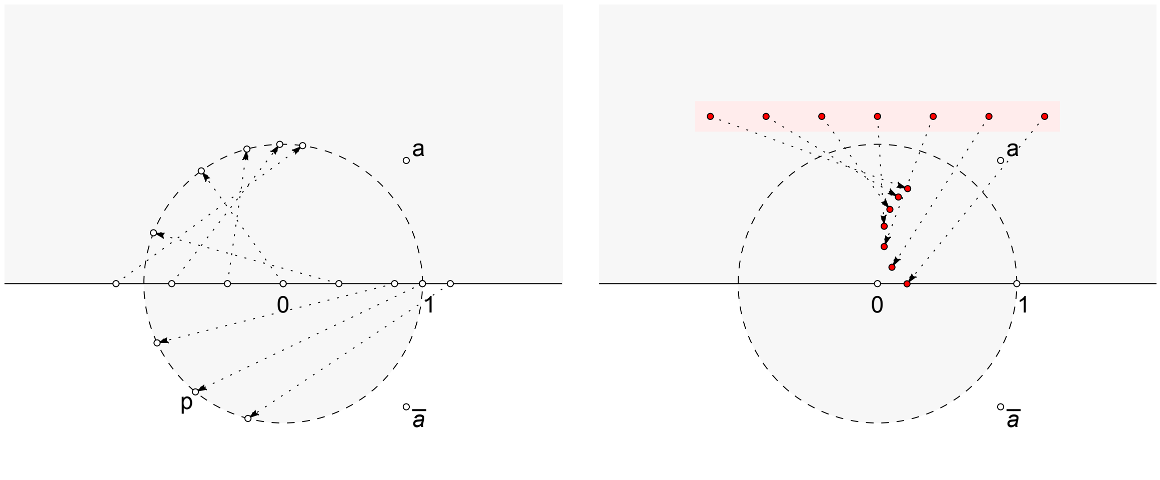
Needham, Chapter 3, Exercise 24, p. 187
G.Palmer, Sept 25, 2021, 1 pm PDT



[a] Mapping real line to C; $\phi=\pi/5$ [b] Mapping horizontal row outside D; $\phi=\pi/5$

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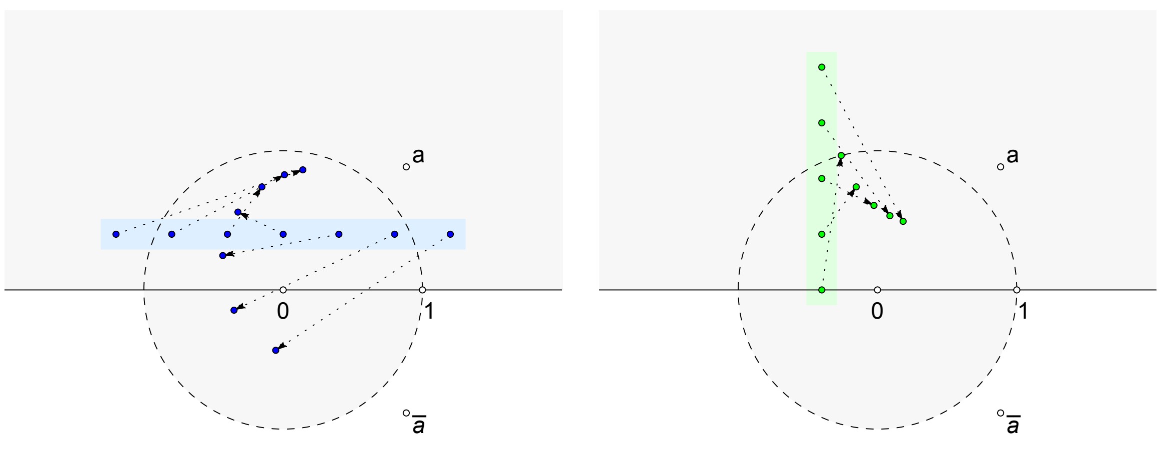

[c] Mapping horizontal row through D; $\phi=\pi/5$ [d] Mapping vertical row through D; $\phi=\pi/5$

`Figure 1. Mapping upper half plane to unit disc`

(24) (i) Use the Symmetry Principle to show that the most general Möbius transformation of the upper half-plane to the unit disc has the form

$$M(z) = e^{i\theta}\left(\frac{z-a}{z-\bar{a}}\right)$$

where Im $a > 0$.

If two points are symmetric with respect to a circle, then their images under a Möbius transformation are *symmetric with respect to the image circle. This is called the "Symmetry Principle"* (p. 148) (underlining added).

It seems best to assume that the formula for M(z) is unknown, even though Needham presented it to us. Knowing only that the goal is to map the UHP to the unit disc, one can see that the UHP is bounded by the real line and the unit disc is bounded by C. From (30), p. 155, we know that given three points on a circle, that circle can be mapped to another in such a way that the region to the left of the first maps to the left of a second. On p. 177 a similar exercise to the current one involves mapping C to itself. The symmetry principle is used there to map one pair of symmetric points in C to another in C. In the current exercise, the symmetry principle could be used to map a pair of points symmetric in the boundary of the UHP to a pair of points symmetric in C. Vasco pointed out (p.c., Forum) that one can think of the UHP as a disc of infinite radius whose boundary is the real line. Since Möbius transformations are conformal, they preserve circles, so such an M would map the real line to C. From (30) it would also map the UHP to the interior of C.

To begin constructing M, one needs a symmetry in the real line (L, a circle of infinite radius). The condition (Im a > 0) limits a to the UHP. Since a and $\overline{a}$ are symmetric in L, a Möbius transformation can be constructed that maps a and $\overline{a}$ to the points 0 and ∞, which are symmetric in C. This is a mapping of symmetries from one circle to another. Figure 1[a] shows that M also maps L to C (figure 1[a]). The Möbius transformation that does this is

$$M_a^0(z) = \frac{z-a}{z-\overline{a}}$$

We apparently have three degrees of freedom to work with. By (49), p. 176, *Möbius automorphisms of D have three degrees of freedom*. The prior sentence on p. 176 appears to define an automorphism of D as a mapping of D to itself. The current exercise is a mapping of the UHP to D, which is not the same, but since we can think of the UHP as a disc with infinite radius, this exercise of mapping UHP to D is very much like mapping D to itself, so three degrees of freedom suffice. The choice of a has used two because a is a complex number with two real coefficients: Re(a) and Im(a). One additional real variable is required to implement the third degree of freedom. Let ϕ be this variable. One can see by inspection of a plot, that $|M_a^0| \leq 1$. Since the domain of z is the upper half plane, the absolute value of the denominator of M_a is always greater than or equal to that of the numerator, so $|M_a^0| < 1$. M_a^0 lands on the unit disc. The third degree of freedom with p = M(1) rotates points of M_a^0, as explained on p. 177.

$$M(z) = k\, M_a(z) = k \left(\frac{z-a}{z-\overline{a}}\right)$$, where k is a constant.

It is required that p = M(1) be a point on C, so

$$1 = |p| = |k| \frac{|1-a|}{|1-\overline{a}|} = |k| \implies k = e^{i\phi}$$

In figure 1 [a-d], the choice of $\theta = \pi/5$ is arbitrary.

We have shown that the most general Möbius transformation of the upper half plane to the unit disc has the form

$$M(z) = e^{i\theta}\left(\frac{z-a}{z-\bar{a}}\right).$$

(ii) The most general Möbius transformation back from the unit disc to the upper half-plane will therefore be the inverse of M(z). Let's call this inverse N(z). Use the matrix form of M to show that

$$N(z) = M^{-1}(z) = \frac{\bar{a}\,z - a\,e^{i\theta}}{z - e^{i\theta}}.$$

Square brackets indicate the matrix form of the Möbius transformation [M].

$$\mathrm{M(z)} = e^{\mathrm{i}\theta}\left(\frac{z-a}{z-\bar{a}}\right) = \left(\frac{e^{\mathrm{i}\theta}\,z-\mathrm{a}e^{\mathrm{i}\theta}}{z-\bar{a}}\right) \equiv \begin{bmatrix} e^{\mathrm{i}\theta} & -\mathrm{a}e^{\mathrm{i}\theta} \\ 1 & -\bar{a} \end{bmatrix}$$

$$[\mathrm{N}] = [M]^{-1} = \begin{bmatrix} -\bar{a} & \mathrm{a}e^{\mathrm{i}\theta} \\ -1 & e^{\mathrm{i}\theta} \end{bmatrix} \equiv \frac{-\bar{a}\,z + \mathrm{a}e^{\mathrm{i}\theta}}{-z + e^{\mathrm{i}\theta}} = \frac{\bar{a}\,z - \mathrm{a}e^{\mathrm{i}\theta}}{z - e^{\mathrm{i}\theta}} = \mathrm{N(z)}\ \ \text{(numerator and denominator times -1)}$$

(iii) Explain why the symmetry principle implies that $N(1/\bar{z}) = \overline{N(z)}$.

For z on the unit disc, z symmetrical to $1/\bar{z}$ by definition. N(z) is in the upper half-plane by the definition of N as the inverse of M, which maps the UPH to the unit disc. N(z) symmetrical to $\overline{N(z)}$ by reflection across real line. N(z) symmetrical to N($1/\bar{z}$) by Principle of Symmetry. Since there are only two possible axes of symmetry in this construction, N($1/\bar{z}$) = $\overline{N(z)}$. See Figure 2 for details regarding two arbitrary choices of z in unit disc.

(iv) Show by direct calculation that the formula for N in part (ii) does indeed satisfy the equation in part (iii).

$$\overline{N(z)} = \mathrm{Conjugate}\left(\frac{\bar{a}\,z - \mathrm{a}e^{\mathrm{i}\theta}}{z - e^{\mathrm{i}\theta}}\right) = \frac{a\,\bar{z} - \bar{a}\,e^{-\mathrm{i}\theta}}{\bar{z} - e^{-\mathrm{i}\theta}} \quad \text{(both z's conjugated)}$$

$$= \frac{e^{\mathrm{i}\theta}\,a\,\bar{z} - e^{\mathrm{i}\theta}\,\bar{a}\,e^{-\mathrm{i}\theta}}{e^{\mathrm{i}\theta}\,\bar{z} - e^{\mathrm{i}\theta}\,e^{-\mathrm{i}\theta}} = \frac{\mathrm{a}e^{\mathrm{i}\theta}\,\bar{z} - \bar{a}}{e^{\mathrm{i}\theta}\,\bar{z} - 1}$$

$$= \frac{\bar{a} - \mathrm{a}e^{\mathrm{i}\theta}\,\bar{z}}{1 - e^{\mathrm{i}\theta}\,\bar{z}} = \frac{\bar{a}(1/\bar{z}) - \mathrm{a}e^{\mathrm{i}\theta}\,\bar{z}(1/\bar{z})}{1\times(1/\bar{z}) - e^{\mathrm{i}\theta}\,\bar{z}(1/\bar{z})} = \frac{\bar{a}(1/\bar{z}) - \mathrm{a}e^{\mathrm{i}\theta}}{1/\bar{z} - e^{\mathrm{i}\theta}} = \mathrm{N}(1/\bar{z})$$

Out[ ]=

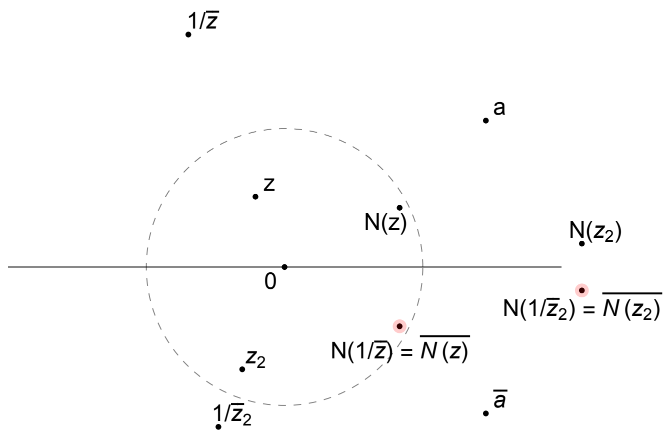

Figure 2. Transformation from unit disc
to upper half−plane by
$N(z) = ((\overline{a}z - ae^{i\theta})/z - e^{i\theta})$, $\theta = 0$