Needham, Chapter 3, Exercise 23, p. 187.
Palmer, September 11, 2021, 5:30 pm PDT

Out[∘]=

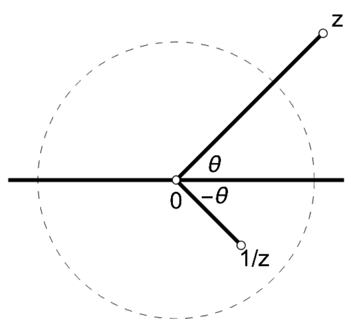

Figure 1. 1/z on $\mathbb{C}$ plane

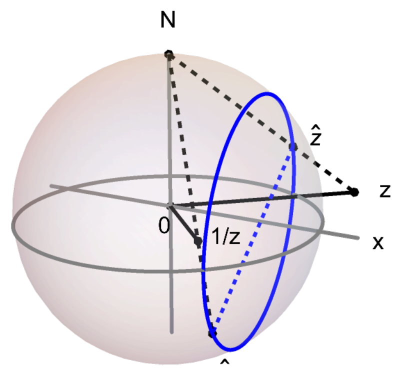

Figure 2. 1/z on Riemann sphere

23. From the simple fact that $z \mapsto (1/z)$ is involutory, deduce that it is elliptic, with multiplier -1.

An involutory transformation maps a complex point to itself with the second application. That is, $M \circ M(z) = z$, so $M \circ M = \mathcal{E}$. An elliptic transformation induces a rotation about the N-S axis of the Riemann sphere. Needham describes $e^{i\alpha}z$ (In the $\mathbb{C}$ plane) as *the simplest archetypal example of a so-called elliptic Möbius transformation* (p. 152). On p. 162, he defines a second complex plane, the $\tilde{z}$-plane, which is a transformation under F of the z-plane on the LHS of [29], p. 162. In this framework, M refers back to the LHS and $\tilde{M}$ refers to a corresponding transformation on the RHS. It is the Möbius transformations on the LHS to which the adjectives *elliptic*, *hyperbolic*, *loxodromic*, and *parabolic* are applied, but it is the RHS to which the multiplier is applied. A key equation is $\tilde{M}(\tilde{z}) = \mathfrak{m}\tilde{z}$, where $\mathfrak{m} = \rho e^{i\alpha}$. Needham states that $\mathfrak{m}$ *not only constitutes a complete description of the mapping $\tilde{M}$ but ... it also completely characterizes the geometric nature of the original Möbius transformation M* [with no tilde]. $\mathfrak{m}$ can be calculated from the fixed points (p. 169), which can themselves be calculated from (27), p. 152.

Since M = 1/z is involutory then so must $\tilde{M}$ be, as angles in the z-plane are duplicated in the $\tilde{z}$-plane. One can write $\mathfrak{m}\tilde{z} \circ \mathfrak{m}\tilde{z} = \mathfrak{m}^2\,\tilde{z} = \tilde{z}$, so $\mathfrak{m}^2 = 1$, and $\mathfrak{m} = \pm 1$, which is not an expansion. Since $\mathcal{E}$ has $\mathfrak{m}$=1, and an involutory transformation is not in general $\mathcal{E}$, though $\mathcal{E}$ is involutory, it must be that, for $\tilde{M}$ other than $\mathcal{E}$, $\mathfrak{m} = -1$. Needham concluded from this that complex inversion is elliptic, and *an infinitesimal neighbourhood of either fixed point is simply rotated about that point through angle π.* How did he conclude that M is elliptic? Since $\mathfrak{m} = -1 = e^{i\pi}$ is a pure rotation, M is elliptic. Since the fixed points are ± 1, the invariant circles center on the x-axis. Figure 2 illustrates the invariant circle through $\hat{z}$ and $(\hat{1/z})$ on the Riemann sphere based on an arbitrary z. This result is consistent with (18), p. 143: *The mapping z*

↦ (1/z) in ℂ induces a rotation of the Riemann sphere about the real axis through an angle of π.

Notes

If M were hyperbolic, under F, the composition of $\rho\tilde{z}$ would map to $\rho^2\,\tilde{z}$, which would be an expansion or contraction, except in the special case of $\rho = 1$. In general, the hyperbolic transformation is non-involutory, so M can not be hyperbolic.

If M were loxodromic, M could not be involutory for the same reason that it could not be hyperbolic. $\tilde{M} \circ \tilde{M}$(z) has a ρ^2 term.

If M were parabolic, $(\tilde{z} + \mathrm{B}) \circ (\tilde{z} + \mathrm{B}) \longrightarrow \tilde{z} + 2\mathrm{B}$, which is non-involutory.

We know that M(z) = 1/z is a Möbius transformation since there are only four possible Möbius transformations, of which the hyperbolic, loxodromic, and parabolic have been shown to be non-involutory, we can conclude that 1/z is elliptic.

But 1/z is not an example of the archetypal elliptic transformation (pp. 152-3). Let z = $re^{i\theta}$. Then 1/z = (1/r) $e^{-i\theta}$, so M implies expansion or contraction as well as a rotation of -2θ (Figure 1). One might expect M to be loxodromic, because M = 1/z seems equivalent to $\left(1/\mid z\mid^2\right) e^{-2\,i\theta}\,z$, but we know it is involutory, so it can't be loxodromic. The blue circle on the Riemann sphere in Figure 2 shows one of the invariant lines of M(z) with rotation about the real line. The radius of the circle reveals a contraction from z. I don't know how this behavior of M is resolved with the definition of the archetypal elliptic Möbius transformation as $e^{i\alpha}\,z$ and the definition of involutory as involving $\mathfrak{m} = e^{i\pi}\tilde{z}$. New definitions would be needed to describe the rotation about the real axis.