Needham, Chapter 2, Exercise 22, p. 187
G. Palmer, September 2, 2021, 9:10 pm

Out[4309]=

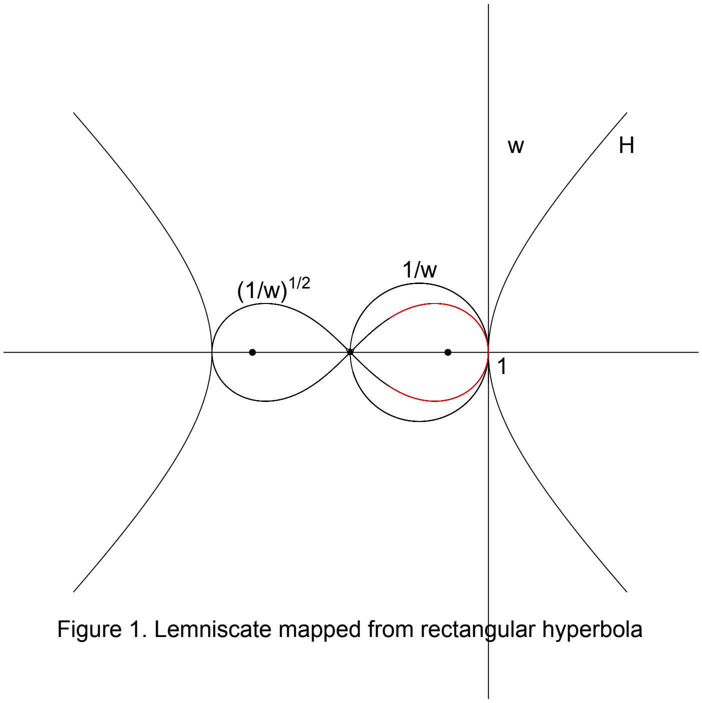

Figure 1. Lemniscate mapped from rectangular hyperbola

22. Let H be the rectangular hyperbola with Cartesian equation $x^2 - y^2 = 1$. Show that $z \mapsto w = z^2$ maps H to the line Re(w) = 1. What is the image of this line under the complex inversion, $w \mapsto (1/w)$? Referring back to figure [9],p. 62, deduce that complex inversion maps H to a lemniscate!
[Hint: Think of inversion as $z \mapsto \sqrt{(1/z^2)}$.]

Rearrange the Cartesian equation, solve for y, and substitute x and y into z = x + i y to obtain an equation for the hyperbola H.

$$y^2 = x^2 - 1$$

$$y = \pm \sqrt{x^2 - 1}$$

$$H = x \pm i \sqrt{x^2 - 1}$$

This is a complex equation for the hyperbola H (Figure 1). Now apply $z \longrightarrow z^2$ to H to obtain the line Re(w) = 1.

$$w(H) = z^2 = \left(x \pm i \sqrt{x^2 - 1}\right)^2 = x^2 \pm 2\,\mathrm{i}x \sqrt{x^2 - 1} - x^2 + 1$$

$= 1 \pm 2\,ix\,\sqrt{x^2 - 1}$

Then if $|x| \geq 1$, $z \longrightarrow w$ maps H to the line Re(w) = 1 (Figure 1). Under inversion 1/w, the vertical line w is mapped to a unit circle centered at (.5) (Figure 1). The point 1 is mapped to 1 and the point ∞ is mapped to 0.

In Figure [9], p. 62, the lemniscate is mapped to a circle by z^2, just as the hyperbola is mapped to a line by z^2. Since the line can be thought of as a circle of infinite radius, the lemniscate can be regarded as a hyperbola. Both the hyperbola and the lemniscate map to circles under z^2. Then the expression $(1/z^2)^{(1/2)} = \sqrt{1/w}$ applied to the circle in Figure 1 plots a lemniscate, though not the same one as shown in [9]. In fact, it plots the lemniscate in Figure 1. The lemniscate in Figure 1 is not congruent with that in [9]. Hence, complex inversion, combined with the additional mappings of the square and square root, maps H to a lemniscate. But in fact, the square and square root are unnecessary, because the square root is the inverse of the square. The hyperbola H maps directly to the lemniscate under complex inversion: $(1/z^2)^{(1/2)} = 1/z$. The red segment of the lemniscate shows this mapping.