Needham, Chapter 3, Exercise 21, pp. 187.

G. Palmer, August 29, 2021, 5:20 pm PDT

21 *(i) Use the matrix representation to show algebraically that the set of Möbius transformations*

$$R(z) = \frac{az+b}{-\overline{b}z+\overline{a}} \quad \textit{with} \qquad |a|^2 + |b|^2 = 1$$

forms a group under the operation of composition.

$$[R_{ab}(z)] = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}$$

The identity matrix is $[\mathcal{E}]$ (p. 157, first •).

Every member of the set represented by the given matrix has an inverse that lies in the set $[R_{ab}(z)]$, because the inverse simply swaps elements 1 and 4 and changes the sign of elements 2 and 3 (left to right, then down):

$$\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{a} & -b \\ \overline{b} & a \end{bmatrix}$$

The conjugate pair of a, $\overline{a}$ is preserved. The conjugate pair with opposite sign of b, $-\overline{b}$ is preserved in -b, $\overline{b}$. These changes preserve the determinant $|a|^2 + |b|^2 = 1$.

Now we need only show that the composite of two such matrices preserves structure and the value of the determinant.

$$[R_{ab}(z)] \circ [R_{cd}(z)] = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\overline{d} & \overline{c} \end{bmatrix}$$

$$= \begin{bmatrix} \text{ac} - b\overline{d} & \text{ad} + b\overline{c} \\ -\overline{\text{ad}} - \overline{b}c & \overline{a}\overline{c} - \overline{b}d \end{bmatrix}$$

On the diagonals, the top left and bottom right elements are conjugates. The top right and bottom left elements are negative conjugates. Therefore structure is preserved. It remains only to show that

$$\left| \text{ac} - b\overline{d} \right| \text{^}2 + \left| \text{ad} + b\overline{c} \right| \text{^}2 = 1$$

One can see by inspection that $|a|$ ^2 + $|b|$ ^2 is the determinant of $[R_{ab}(z)]$ and $|c|$ ^2 + $|d|$ ^2 is the det $[R_{cd}(z)]$. Since these = 1, then from p. 159,

$$|ac - b\bar{d}|^2 + |ad + b\bar{c}|^2 = \det([R_{ab}(z)].[R_{cd}(z)])$$

$$= \det([R_{ab}(z)]) \det([R_{cd}(z)]) = 1.$$

The given matrix $[R_{ab}(z)]$ passes the identity test, the inverse test, and the composition test. It therefore forms a group under composition.

(ii) Using the interpretation of these functions given on p. 162, explain part (i) geometrically.

In part (i), the given matrix is the matrix of the most general rotation of the Riemann sphere (p. 162, (38). Then $[R_{ab}(z)].[R_{cd}(z)]$ is a rotation followed by a rotation. Since we have shown algebraically in part (i) that the composition results in a new matrix of the same structure, we have shown that the new matrix is also a way to express the most general rotation of the Riemann sphere and that these rotations form a group under the operation of composition. The function of the inverse matrix is to impose a rotation in the opposite direction from the given matrix.