Needham, Chapter 3, Exercise 19, Part iii
G. Palmer, August 16, 2:50 pm PDT

Out[32]=

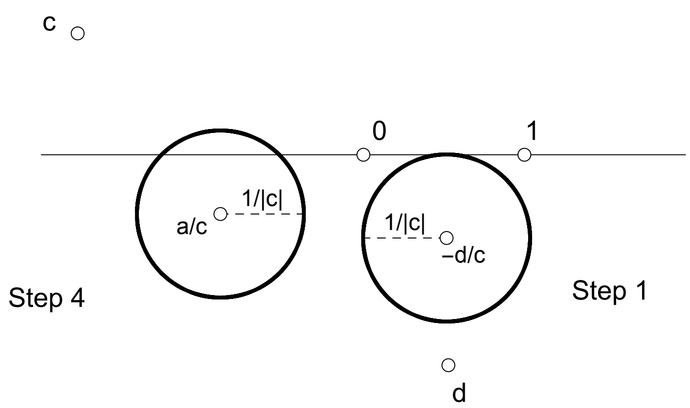

Figure 2. The isometric circle of M

(iii) Hence show geometrically that the circle $\mathcal{I}_M$ with equation $|cz + d| = 1$ is mapped by M to a circle of equal size. Furthermore, show that each arc of $\mathcal{I}_M$ is mapped to an image arc of equal size. For this reason, $\mathcal{I}_M$ is called the isometric circle of M.

$|cz + d| = 1$ can be written $|z + d/c| = 1/|c|$. Since $|z-p| = r$ is a circle centred at p, it is evident that the original equation describes a circle of radius $1/|c|$ centred at $(-d/c)$. Then, follow the steps in (3), p. 124). In each step, the result will be a short form of the equation using one of the transformed z_i and a long form in which the equation is rewritten by substituting an expression in z_0 for z_i.

Step 1

$z \longrightarrow z + d/c$ translates the circle by d/c. The RHS of the equation is not affected because the translation doesn't change the radius. The transformation creates a new circle, so the equation must change. Consider the canonical complex equation for a circle $|z-p| = r$. To move p to the origin, write

$$|(z - p) + p)| = r$$

$$|z - 0| = r$$

This is what we wish to accomplish with step 1, so that subsequent transformations in steps 2 and 3 will be symmetrical about the origin.

Rewrite the initial equation as

$$|z_0 + d/c| = 1/|c| \qquad (1)$$

Let $z_1 = z_0 + d/c$. Now comes the part that I found most difficult to follow. Vasco noticed that the equation can be rearranged as $z_0 = z_1 - d/c$ and this result can be substituted into (1).

$$| (z_1 - d/c) + d/c) | = 1/ | c |$$

$$| z_1 | = 1/ | c |$$

This is a circle of radius 1/|c| centred at the origin.

Step 2: z ⟶ 1/z

Step 2 inverts the circle in the unit circle; the radius is inverted and there is a rotation of $-2\theta + \pi$ about the origin. The centre of the circle remains where it is. Since the radius depends on what happens to z, we apply the inversion to both sides.

$$| 1/z_1| = |c| \qquad (2)$$

Long form: $| 1 / (z_0 + d / c) | = | c |$

Step 3: $z \longrightarrow (-1/c^2)$

Step 3 amplifies (or shrinks) the radius and rotates the circle by $(-2\phi + \pi)$, where ϕ = Arg[c]. The radius is returned to its original size. The centre remains at the origin. Since the rotation is about the origin, one may regard it as mapping the circle to itself prior to or following the amplification. Since the radius depends on z, it is also multiplied by $| -1/c^2 |$, which is applied on both the LHS and the RHS of the equation. Let $z_2 = 1/z_1$.

$| z_2 | = | c |$ by substitution into (2)

$| -1/c^2 || z_2 | = | -1/c^2 || c |$ multiplication through by $| -1/c^2) |$

$$| (-1/c^2) z_2 | = 1/ | c | \qquad (3)$$

Long form: $| (-1/c^2) (1/(z_0 + d/c)) | = 1/ | c |$

Step 4: z ⟶ z + a/c

Step 4 translates the centre by a/c without affecting the radius. Let $z_3 = \longrightarrow (-1/c^2) z_2$.

$| z_3 | = 1/ | c |$ by substitution into (3)

$| z_3 + a/c | = 1/ | c |$ addition of a/c

Long form: $|\,(-1/c^2)\,(1/(z_0 + d/c)) + a/c\,| = 1/\,|\,c\,|$ (4)

This should be a circle centred at (–a/c) with the same radius 1/|c| as the circle in the equation $|cz + d| = 1$. We know empirically by applying M(z) = (a z + b)/(c z + d) that the new circle is centred at a/c. The code which plotted figure 2 was a one-to-one translation of M(z). Does the long form resolve to $M(z_0) = \frac{az_0+b}{cz_0+d}$? Let $z_4 = (-1/c^2)\,(1/(z_0 + d/c)) + a/c$, so (4) becomes

$|\,z_4\,| = 1/\,|\,c\,|$ by substitution into (4)

Then

$$\left(-1/c^2\right)\,(1/(z_0 + d/c)) + a/c = a/c - 1/[c\,(cz_0 + d)]$$

$$= \frac{a(cz_0+d)-1}{c(cz_0+d)} = \frac{acz_0+ad-1}{c(cz_0+d)} = \frac{acz_0+bc}{c(cz_0+d)}$$ (because (ad - 1) = bc when M(z) is normalized)

$$= \frac{az_0+b}{cz_0+d} = M(z_0)$$

Then $|\,z_4\,| = 1/\,|\,c\,|$ becomes $|\,M(z_0)\,| = 1/\,|\,c\,|$. This is what we wanted. We note also that if the new circle is to have the form |p–r| = 1/|c|, then we would have to write

$$\left|\,\frac{1}{c(cz_0+d)} - \frac{a}{c}\,\right| = 1/\,|\,c\,|,\quad \text{where } p = \frac{1}{c(cz_0+d)} \text{ and } r = a/c.$$

In this application of M, z_0 from $|\,z_0 + d/c\,| = 1/\,|\,c\,|$ was successively mapped to z_1, z_2, z_3, and z_4. Each step of the decomposition creates a new circle and a new equation.

Furthermore, show that each arc of $\mathcal{I}_M$ is mapped to an image arc of equal size. For this reason, $\mathcal{I}_M$ is called the isometric circle of M.

An arc of $\mathcal{I}_M$ can only map to a different size if it is subjected to a different rotation at different positions on the circle, or if the size of the radius changes between the image and the preimage. We have shown that the radius is constant under M. M subjects z to rotation in steps (ii) and (iii). At every point of z on the image circle, the rotation applied by step (ii) z = 1/z is $e^{-i2\theta}$. It does not matter where the arc of angle $\Delta\theta = \theta_1 - \theta_0$ in the pre-image begins and ends. All points of the arc in the pre-image are subjected to negative rotation by the angle (-2θ), which reflects θ in the real line. The size ($s = r\,|\Delta\theta|$) of the arc is inverted under this step, because the radius is inverted. In step (iii) z is subject to the rotation of –1/c^2. Since c is a constant, if ϕ is the angle of c, the angle of rotation applied to any point of z is $(-2\phi + \pi)$. The radius is restored to its original size, and size of an arc ($s = r\,|\Delta\theta|$) depends only on $\Delta\theta$, so arc size in the pre-image is preserved under M.