Needham, Chapter 3, Exercise 17, p. 186.
G. Palmer, August 8, 2021, 9:30 pm PDT

Out[873]=

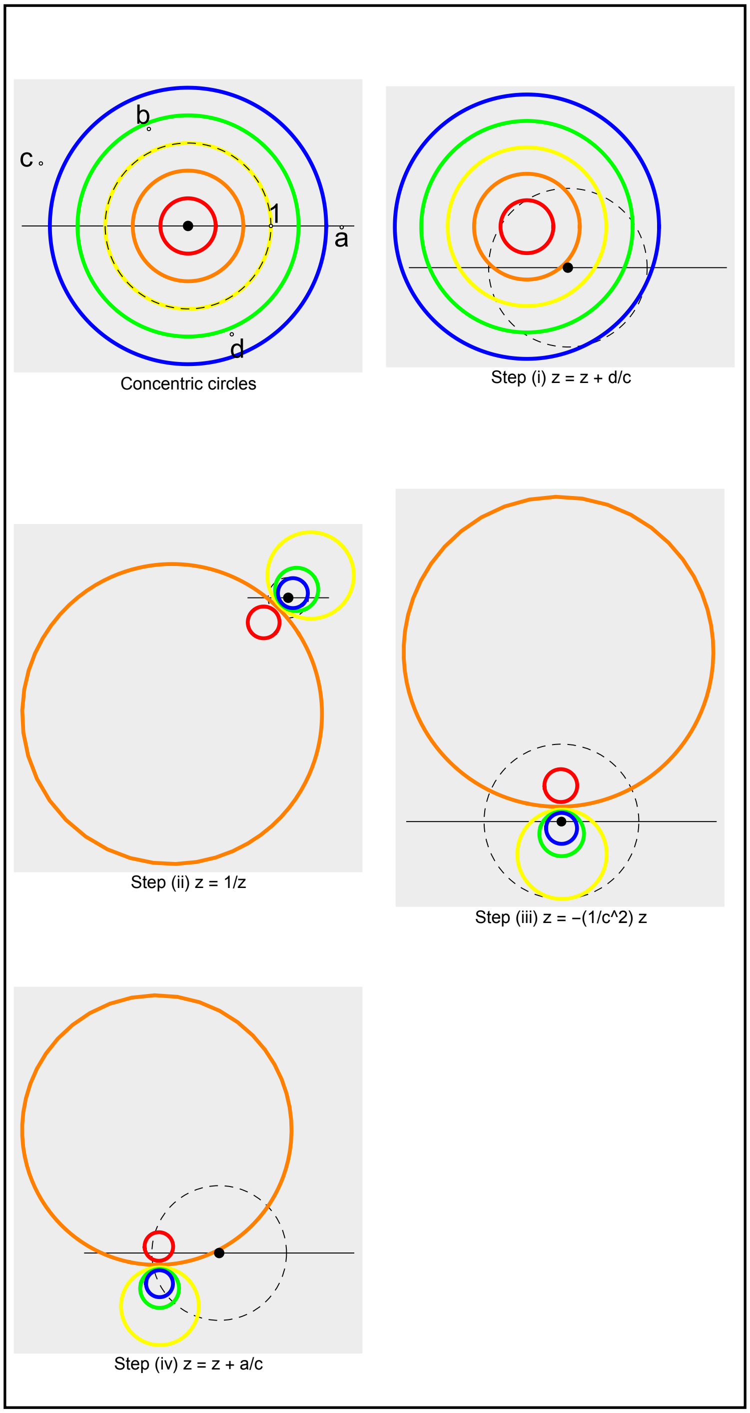

Figure 1. Successive steps of M
on family of concentric circles

19 Let $M(z) = \frac{az+b}{cz+d}$ be normalized.

(i) Using (3) [p. 124]*, draw diagrams to illustrate the successive effects of these transformations on a family of concentric circles. Note that the image circles are generally not concentric.*

See Figure 1. Note that centring the concentric family on the origin as in Figure 1 may give the appearance of a special case, but for this exercise, the origin is no different from any other point. The point –(d/c) would be a special case, as we learn in part (ii). Step (iii) is labeled "–(1/c^2) z" rather than "(–(ad–bc)/c^2) z" because the coefficients of M are normalized. The black dotted circle centred on the black dot is the unit circle centred on the origin.

Step (i) translates all circles by d/c.
Step (ii) sends the green and blue circles from the outside to the inside of the transformed yellow unit circle and it sends the red and orange circles from the inside to the outside.
Step (iii) shrinks the radii of the circles by |c|^2 and rotates them by Arg(c) + π.
Step (iv) translates the circles by a/c.

(ii) Deduce that the image circles are concentric if and only if the original family of circles are centred at q = –(d/c). Write down the centre of the circles in this case. [Note that that is not the image of the centre of the original circles: M(q) is the point at infinity!].

If the original family of circles are centred at q = –(d/c), the translation in step (i) centres the family at the origin, still concentric. Step (ii) inverts them in the unit circle, but they are still concentric. Step (iii) shrinks their radii and rotates them by Arg(c) + π, but the rotation about the origin simply sends each circle to itself, so we see no change. They remain concentric. Step (iv) translates the centre of the family to (a/c). They remain concentric. This shows that they remain concentric *if the original center is –(d/c)*.

If the original centre is not –(d/c), then step (i) moves them all to a location where they are centred off the origin, but still concentric. Step (ii) inverts them in the unit circle centred at the origin (Figure 1). This breaks their concentricity because inversion about the yellow circle is not the same. Step (iii) rotates the family about the origin, not about their centres, and it shrinks the radii, but it does not repair the broken concentricity. Step (iv) simply translates the non-concentric family by (a/c). Hence, *the image circles are concentric if and only if the original family of circles are centred at q = –(d/c)*.