Needham, Chapter 3, Exercise 18, p. 186.
G. Palmer, Aug 3, 2021, 12:15 pm PDT

Out[2824]=

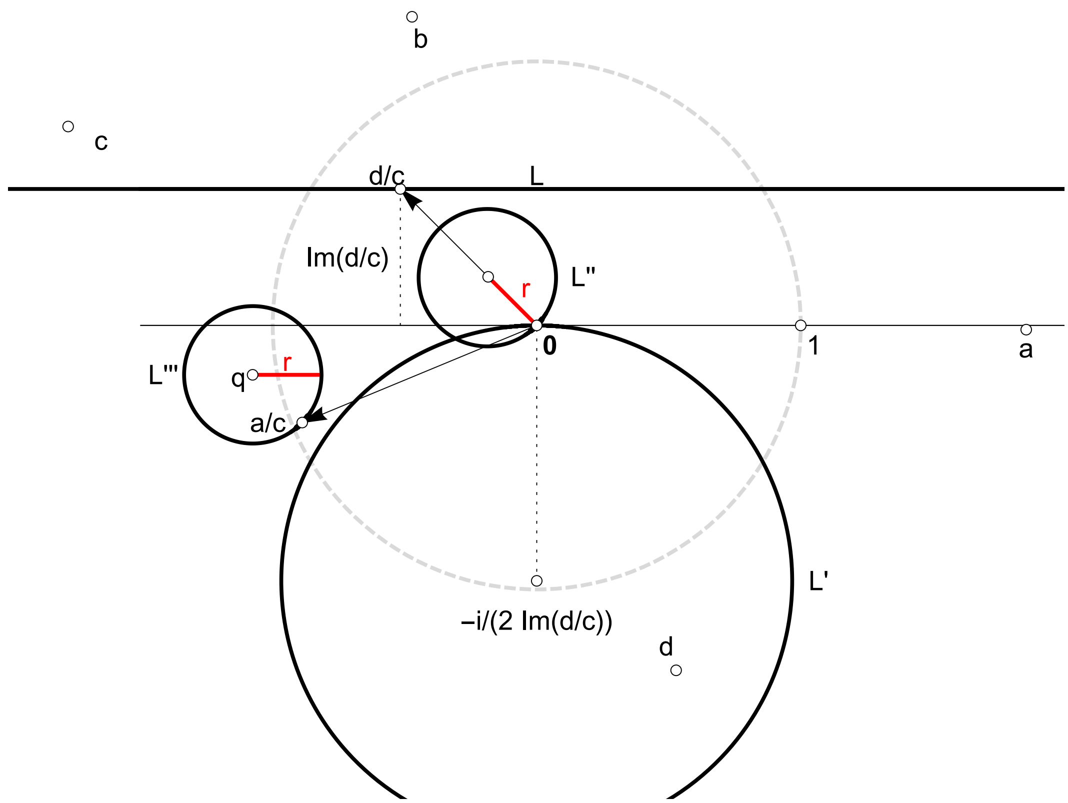


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Figure 1. Möbius transformation of L
and calculation of κ of L'''
with four arbitrary points a, b, c, d
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18 The curvature κ of a circle is defined to be the reciprocal of its radius. Let $M(z) = \frac{az+b}{cz+d}$ be normalized. Use (3) to show geometrically that M maps the real line to a circle of curvature

$$\kappa = \left| 2c^2 \, Im\left(\frac{d}{c}\right) \right|.$$

A modified version of the decomposition (3) on p. 124.

1) $zi \longrightarrow z + d/c$
2) $zii \longrightarrow 1/zi \qquad = 1/(z + d/c)$
3) $ziii \longrightarrow (-(ad - bc)/c^2)/(z+d/c)$
4) $ziv \longrightarrow$ -(ad-bc)/(c^2 (z+d/c)) + a/c

The values of a, b, c, and d are arbitrary and normalized with $1/\sqrt{ad - bc}$. Values of z lie on the real line. Treating the real line as a curve, one can construct a step by step derivation of r and subsequently $\kappa = 1/r$. The light gray dashed circle is the unit circle.

Step 1 translates the real line by d/c, mapping the real line to the horizontal line L at height h = Im(d/c).

Step 2 is a complex inversion in the unit circle which maps L to the circle L’. The minimal point of L’ on the y axis can be calculated as –1/(h i) = –1/(i Im(d/c)). The radius of L’ is (1/2) |1/Im(d/c)|.

Step 3 is an expansion plus a rotation. Since normalization ensures that (ad - bc) = 1, the step reduces to -1/c^2, which itself is both an expansion and a rotation mapping L’ to the circle L’’. The rotation is (2 Arg(c) + π) and the expansion, really a contraction in this case, is 1/|c|^2. The radius of the circle contracts by the same factor. Hence, the radius can now be written as (1/2)|(1/c^2)(1/ Im(d/c))|. Since the curvature is the reciprocal of the radius, one can write

κ = |2 c^2 Im(d/c)|

Step 4 is another translation which maps L’’ to L’’’. In a Newtonian framework, the translation has no effect on the radius or the curvature.

Figure 2 presents a check on the derivation by applying the steps to the point z = 1. The circle L’ was obtained by applying M(z) to points in L. The black arrows are not intended to show vectors. At each step, they point to the next step. As a check on zii, the conjugate of zii falls on the extended line through zi, as it should. z is a point on L, so ziv is a point on L’ (Figure 2 has only circles L, L’, and the unit circle). The point M(1) falls exactly on ziv, as it should, providing evidence that L’ has the correct radius and that the steps of decomposition (3) are consistent with M(z). The radius of L’ matches the the reciprocal r = 1/κ obtained with the given formula for κ. This shows geometrically that *M maps the real line to a circle of curvature*

$$\kappa = \left| 2 c^2 Im\left(\frac{d}{c}\right) \right|.$$

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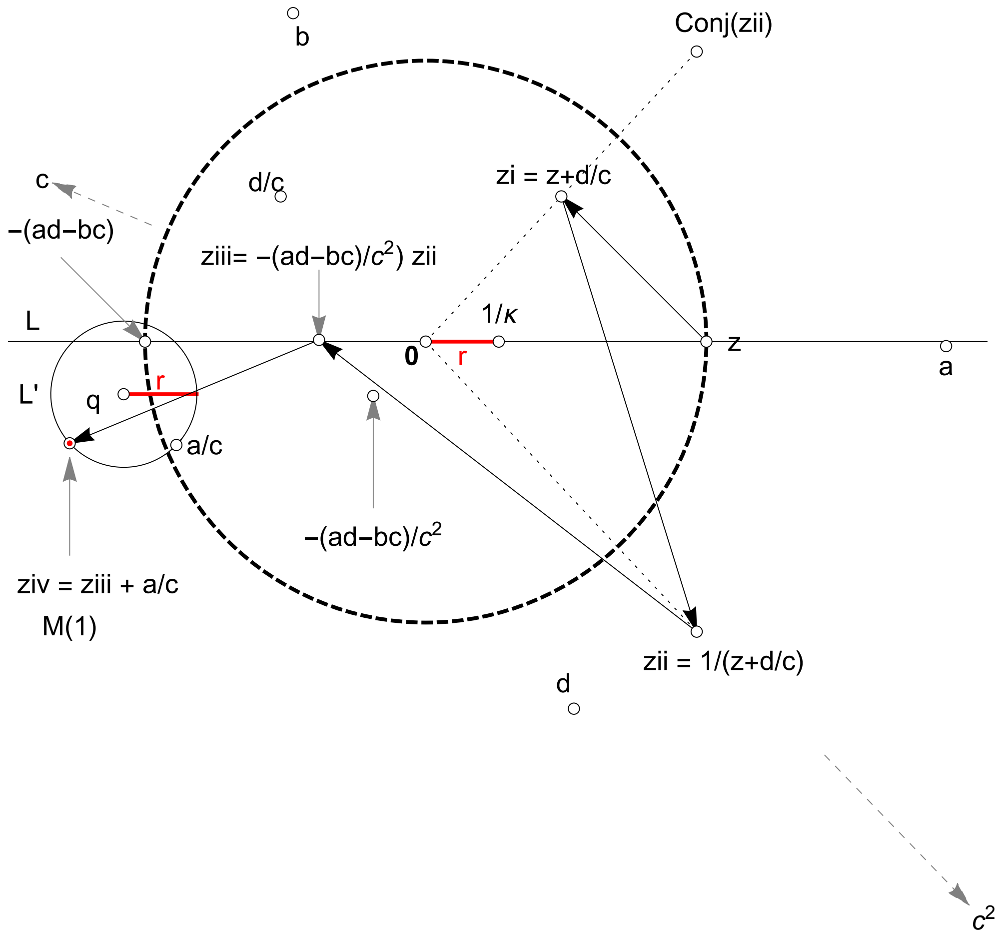


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Figure 2. Möbius transformation of L
and calculation of κ of L'
with four arbitrary points a, b, c, d
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